ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Pures Appl. 86 (2006) 427–447
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Solvability in the large for a class of vector fields on the torus

A.P. Bergamasco [a,∗,1], P.L. Dattori da Silva [b,2]

[a] *Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Caixa Postal 668, CEP 13560-970, São Carlos, SP, Brazil*

[b] *Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, CEP 13565-905, São Carlos, SP, Brazil*

Received 16 March 2006

Available online 18 September 2006

Abstract

We study a class of complex vector fields defined on the two-torus of the form $\mathsf{L} = \partial/\partial t + (a(x,t) + ib(x,t))\partial/\partial x$, $a, b \in C^\infty(\mathbb{T}^2;\mathbb{R})$, $b \not\equiv 0$. We view L as an operator acting on smooth functions and present conditions for L to have either a closed range or a finite-codimensional range. Our results involve, besides condition $(\mathcal{P})$ of Nirenberg and Treves, the behavior of $a + ib$ near each one-dimensional Sussmann orbit homotopic to the unit circle. One of the main goals of our work is to provide some clarification about the role played by the coefficient a in the validity of the above properties of the range.

Résumé

On étudie une classe de champs de vecteurs complexes definis sur le tore et de la forme $\mathsf{L} = \partial/\partial t + (a(x,t) + ib(x,t))\partial/\partial x$, $a, b \in C^\infty(\mathbb{T}^2;\mathbb{R})$, $b \not\equiv 0$. On considère L comme un opérateur defini sur les fonctions indéfiniment différentiables et on donne des conditions pour que l'image de L soit fermée et pour qu'elle soit aussi de codimension finie. Nos résultats utilisent la condition $(\mathcal{P})$ de Nirenberg et Trèves ainsi que le comportement de $a + ib$ près de chaque orbite de Sussmann unidimensionnelle homotope au cercle unitaire. Un des buts principaux de notre article est de préciser le rôle joué par le coefficient a dans la validité des proprietés de l'image ci-dessus.

MSC: primary 35A05; secondary 58Gxx

Keywords: Global solvability; Complex vector fields; Condition $(\mathcal{P})$; Sussmann orbits; Propagation of singularities; Bicharacteristics

1. Introduction

The *local* solvability of a linear partial differential operator of principal type is characterized by the well-known *Nirenberg–Treves* condition $(\mathcal{P})$ (see [14]).

∗ Corresponding author.
E-mail addresses: apbergam@icmc.usp.br (A.P. Bergamasco), dattori@dm.ufscar.br (P.L. Dattori da Silva).
1 Partially supported by CNPq and FAPESP.
2 Supported by FAPESP.

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2006.08.001

In this work we are interested in studying the *global* solvability of a class of non-singular complex vector fields on the two-torus $\mathbb{T}^2_{(x,t)} \simeq \mathbb{R}^2/2\pi\mathbb{Z}^2$ that satisfy condition $(\mathcal{P})$. This class consists of vector fields of the form:

$$\mathsf{L} = \partial/\partial t + \big(a(x,t) + \mathrm{i}b(x,t)\big)\partial/\partial x, \quad a, b \in C^\infty\big(\mathbb{T}^2; \mathbb{R}\big),\ b \not\equiv 0; \tag{1.1}$$

thus our vector fields are truly non-real.

The vector field L is said to be *globally solvable* if the range of the operator $\mathsf{L}: C^\infty(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2)$ is closed; if, furthermore, $\mathsf{L}(C^\infty(\mathbb{T}^2))$ has finite codimension then L is said to be *strongly solvable*.

For vector fields such as (1.1), condition $(\mathcal{P})$ has a simple statement: L satisfies condition $(\mathcal{P})$ if and only if the function b does not change sign along the integral curves of $\partial/\partial t + a(x,t)\partial/\partial x$ (see [13], Theorem 3.7).

An orbit (in the sense of Sussmann [21]; see also [6,23]) of a non-singular complex vector field, L, defined on a smooth two-dimensional manifold X is an equivalence class with respect to the following relation: two points $x, y \in X$ are said to be equivalent if there is a finite number of integral curves of $\pm\Re L$ and $\pm\Im L$ such that their juxtaposition connects x and y. If $\mathcal{B}$ is an orbit and there is $x \in \mathcal{B}$ such $\Re L(x)$ and $\Im L(x)$ are linearly independent then $\mathcal{B}$ is two-dimensional; otherwise, $\mathcal{B}$ is one-dimensional.

For vector fields of the form (1.1) the only possibility for $\Re\mathsf{L}$ and $\Im\mathsf{L}$ to be linearly dependent at $p = (x_0, t_0)$ is if $b(x_0, t_0) = 0$; hence if p belongs to a one-dimensional orbit, γ, then $\Re\mathsf{L}(p)$ is tangent to γ and $\Im\mathsf{L}(p) = 0$. It follows that γ is a one-dimensional orbit of L if and only if γ is an integral curve of $\Re\mathsf{L}$ along which $\Im\mathsf{L} = 0$.

It is convenient to divide the study according to the different possibilities for the integral curves of $\Re\mathsf{L}$ and for the geometry of the orbits. As a first case, assume that each integral curve of $\Re\mathsf{L}$ is periodic; then it is possible to transform our vector field into a new vector field whose real part is equal to $\partial/\partial t$, a case which is already treated in [1]. As a second case, assume that L has no periodic one-dimensional Sussmann orbits. In this case, we can prove that (see Section 3) $\mathbb{T}^2$ is the only orbit; thus condition (i) of Theorem 7.1 of [12] is satisfied and hence L is strongly solvable.

Hence we have the right to restrict ourselves to the case where L has at least one periodic, one-dimensional Sussmann orbit and $\Re\mathsf{L}$ has at least one non-periodic integral curve.

Before we state our results we will describe some of the known facts.

When $a \equiv 0$, the properties of interest here have been studied in [1]: if $\mathsf{L} = \partial/\partial t + \mathrm{i}b(x,t)\partial/\partial x$ satisfies condition $(\mathcal{P})$ and if, for each x such that $t \mapsto b(x,t) \equiv 0$ the function b vanishes of order $m = m(x) \geqslant 2$, then L is strongly solvable. Moreover, a necessary condition for global solvability is that b vanish of finite order at $\{x\} \times \mathbb{T}^1$.

The special class where a and b are independent of t has been dealt with in [5]. It was proved there that the finiteness of the order of zeros of $a + \mathrm{i}b$ is necessary for strong solvability. Furthermore, the interplay between the order of vanishing of common zeros of a, and of b, is crucial for the strong solvability.

When a and b are independent of x (the *tube case*) the problem of global solvability has been studied in [15] (however no strong solvability was considered there).

Now we will describe our results. Let $\mathcal{K}$ be the union of all one-dimensional, *periodic* orbits of the structure defined by L. In other words, let $\mathcal{K}$ be the union of all periodic integral curves of $\Re\mathsf{L}$ along which $b \equiv 0$ (see Subsection 3.1). By our assumptions we have $\mathcal{K} \neq \emptyset$, and, as consequence of $b \not\equiv 0$, we have $\partial\mathcal{K} \neq \emptyset$.

Concerning necessary conditions for global solvability we have the following result: if L is globally solvable then, for each $x_0 \in \mathbb{T}^1$ such that $\{x_0\} \times \mathbb{T}^1$ is contained in $\partial\mathcal{K}$, the function $a + \mathrm{i}b$ cannot be flat at $\{x_0\} \times \mathbb{T}^1$ (Theorem 2.4).

As far as sufficient conditions are concerned, we will work (as in [1,5]) under the assumption that $\mathcal{K}$ consists of a finite number of periodic one-dimensional orbits; in other words, for some $q \in \mathbb{Z}_+$, $\mathcal{K} = \bigcup_{k=1}^q \gamma_k$. After a change of coordinates (see [8,11]) we can assume $\mathcal{K} = \mathcal{N} \times \mathbb{T}^1$, where $\mathcal{N} = \{x \in \mathbb{T}^1; a(x,t) + \mathrm{i}b(x,t) = 0, \forall t \in \mathbb{T}^1\} = \{x_1, \dots, x_q\}$. Suppose that condition $(\mathcal{P})$ is satisfied and, for each $x_k \in \mathcal{N}$, there is a neighborhood, $\mathcal{V}_k$, such that $a(x,t) + \mathrm{i}b(x,t) = (x - x_k)^{m_k}(a_0(x,t) + \mathrm{i}b_0(x,t))$ in $\mathcal{V}_k$. Under the hypotheses above, if $t \mapsto b_0(x_k, t) \not\equiv 0$ and $m_k \geqslant 2$, for each $k = 1, \dots, q$, then L is strongly solvable (Theorem 3.4).

As in [1] we use, in an essential way, the so-called *method of descent* (see also [9,10]) in order to transform our vector field, L, into a new vector field $\mathcal{L}$, with more variables but less degeneracy, to which the results of Hörmander can be applied to yield useful information. One of the main technical difficulties that had to be overcome was the construction of an appropriate semi-global first integral Z for $\mathcal{L}$ near a one-dimensional orbit. In the end we also use a result (from [1]) concerning the existence of a flat solution (for an abstract equation in Hilbert space), when the right-hand side is flat.

The solvability properties under study are clearly invariant under multiplication by a non-vanishing complex-valued function; our class of vector fields is however not invariant. On the other hand, we hope that our results will lead to such an invariant formulation.

2. A necessary condition for global solvability

As explained in the introduction if L, given by (1.1), satisfies condition $(\mathcal{P})$ and L has no periodic, one-dimensional Sussmann orbit then L is strongly solvable. In this section, we assume that L satisfies condition $(\mathcal{P})$ and that L has at least one periodic one-dimensional Sussmann orbit. After a unimodular transformation, if necessary, we can assume, as in [4], that such an orbit is homotopic to $\{0\} \times \mathbb{T}^1$. We will present a necessary condition for the global solvability of the vector field L; in particular, it will also be a necessary condition for the strong solvability of L. Let $\mathcal{K}$ be the union of all one-dimensional, periodic Sussmann orbits of the structure defined by L. In other words, let $\mathcal{K}$ be the union of all periodic integral curves of $\Re\mathsf{L}$ along which $b \equiv 0$. By assumption, we have $\mathcal{K} \neq \emptyset$, and, since $b \not\equiv 0$ also $\partial\mathcal{K} \neq \emptyset$. The necessary condition is as follows: under the above assumptions, if $x_0 \in \mathbb{T}^1$ is such that $\{x_0\} \times \mathbb{T}^1 \subset \partial\mathcal{K}$, then $a + \mathrm{i}b$ is not flat along $\{x_0\} \times \mathbb{T}^1$ (see Theorem 2.4).

Our proof of the necessity follows the classical path (see [12]): the assumption of global solvability implies certain inequalities, whereas the flatness of $a + \mathrm{i}b$ implies that such inequalities are violated.

As in [1] we will use the following characterization (due to [19], p. 10) of operators having closed range:

Lemma 2.1. (Köthe) *Let E, F be Fréchet spaces, with F separable. Then a continuous linear mapping $A : E \to F$ has closed range if and only if the following property holds: given $x_n \in F'$ with ${}^tA(x_n) \to 0$ weakly, there exists $y_n \in F'$ with $y_n \to 0$ weakly such that ${}^tA(y_n) = {}^tA(x_n)$.*

Lemma 2.2. *If L is globally solvable then given any sequence $\{\mu_n\}$ in $\mathcal{D}'(\mathbb{T}^2)$, such that ${}^t\mathsf{L}\mu_n \to 0$ weakly, there exist $C > 0$ and $N \in \mathbb{Z}_+$ such that*

$$\big|\mu_n\big[(a+\mathrm{i}b)\tilde{\varphi}\big]\big| \leqslant C\big\|(a+\mathrm{i}b)\tilde{\varphi}\big\|_{(N)}, \quad \forall\varphi \in C^\infty\big(\mathbb{T}^1\big),\ \int \varphi = 0,\ \forall n \in \mathbb{Z}_+, \tag{2.1}$$

where $\tilde{\varphi} \in C^\infty(\mathbb{T}^2)$ is defined by $\tilde{\varphi} \doteq \varphi \otimes 1_t$.

The proof of this lemma will be omitted because it is a simple adaptation of the one of Lemma 2.1 in [1].

Lemma 2.3. *Let $\mathcal{K}$ be the union of all one-dimensional, periodic Sussmann orbits of the structure defined by L. Suppose that $x_0 \in \mathbb{T}^1$ is such that $\{x_0\} \times \mathbb{T}^1 \subset \partial\mathcal{K}$. Then (at least) one of the following possibilities must occur:*

($\star$.1) *each open neighborhood, V, of $\{x_0\} \times \mathbb{T}^1$ contains a periodic integral curve, γ, of $\Re\mathsf{L}$, such that $b \circ \gamma \not\equiv 0$;*
($\star$.2) *for each open neighborhood V, of $\{x_0\} \times \mathbb{T}^1$, the set $V \setminus (\{x_0\} \times \mathbb{T}^1)$ contains a periodic integral curve, γ, of $\Re\mathsf{L}$, with γ equal to the ω-limit set (or to the α-limit set) of some non-periodic integral curve of $\Re\mathsf{L}$;*
($\star$.3) *there is a non-periodic integral curve of $\Re\mathsf{L}$, whose ω-limit set (or whose α-limit set) is equal to $\{x_0\} \times \mathbb{T}^1$.*

Proof. Assume that ($\star$.3) does not hold. Then there is an open neighborhood, W, of $\{x_0\} \times \mathbb{T}^1$ such that each integral curve of $\Re\mathsf{L}$ through a point in $W \setminus (\{x_0\} \times \mathbb{T}^1)$ is either periodic or else has periodic integral curves as its ω-limit and α-limit sets. In particular, any open neighborhood of $\{x_0\} \times \mathbb{T}^1$ contained in W, contains infinitely many periodic integral curves of $\Re\mathsf{L}$. In this situation, if we also have that ($\star$.2) does not hold, then there is an open V which is foliated by periodic integral curves of $\Re\mathsf{L}$. Now, since $\{x_0\} \times \mathbb{T}^1 \subset \partial\mathcal{K}$, it follows that each open neighborhood of $\{x_0\} \times \mathbb{T}^1$ contains infinitely many periodic integral curves of $\Re\mathsf{L}$ which are not one-dimensional orbits; in other words, ($\star$.1) holds. The proof is complete. □

Theorem 2.4. *Consider the vector field,*

$$\mathsf{L} = \partial/\partial t + \big(a(x,t) + \mathrm{i}b(x,t)\big)\partial/\partial x, \quad a, b \in C^\infty\big(\mathbb{T}^2; \mathbb{R}\big),\ b \not\equiv 0,$$

on the two-torus $\mathbb{T}^2$*, and assume that* L *satisfies condition* $(\mathcal{P})$*. Let* $\mathcal{K}$ *be the union of all one-dimensional, periodic Sussmann orbits of the structure defined by* L*. If* L *is globally solvable, then for every* $x_0 \in \mathbb{T}^1$ *such that* $\{x_0\} \times \mathbb{T}^1 \subset \partial\mathcal{K}$*, the function* $a + \mathrm{i}b$ *is not flat along* $\{x_0\} \times \mathbb{T}^1$*.*

Proof. We proceed by contradiction and assume that $a + \mathrm{i}b$ is flat along $\{x_0\} \times \mathbb{T}^1$. Our proof will deal separately with each of the three cases in Lemma 2.3, and in each case we will assume that the curves appearing in each case lie in the region $x > x_0$ (if they lie in the region $x < x_0$ the proof is analogous). Before dealing with the three cases we will make some general remarks.

Let $\varphi \in C_0^\infty((-1,1))$ such that $\varphi \equiv 1$ on the interval $[0, 3/4]$, $\varphi \geqslant 0$ on $[-1/2, 1]$, $\varphi \leqslant 0$ on $[-1, -1/2]$, and $\int \varphi = 0$. Consider a strictly decreasing sequence (x_n) which converges to x_0 and satisfies $x_n - x_0 < \frac{2}{3}$. Define the following sequence of functions:

$$\varphi_n(x) = \varphi\left(\frac{2}{3}\frac{x - x_0}{x_n - x_0}\right), \quad n \in \mathbb{Z}_+. \tag{2.2}$$

Note that $-1 < \frac{2}{3}\frac{x-x_0}{x_n-x_0} < 1 \Leftrightarrow \frac{5x_0-3x_n}{2} < x < \frac{3x_n-x_0}{2}$. Hence $\operatorname{supp}(\varphi_n) \subset (\frac{5x_0-3x_n}{2}, \frac{3x_n-x_0}{2})$. Also, $\frac{5x_0-3x_n}{2} < x_0 < x_n < \frac{3x_n-x_0}{2}$ and $\varphi_n \equiv 1$ on $[x_0, \frac{9x_n-x_0}{8}]$. Since $a = O(|x - x_0|^2)$, we can find a neighborhood $\mathcal{U} \times \mathbb{T}^1$ of $\{x_0\} \times \mathbb{T}^1$ such that, for every $(x,t) \in \mathcal{U} \times \mathbb{T}^1$ we have $|a(x,t)| < \frac{1}{16\pi C'}|x - x_0|$; here C' is the Lipschitz constant of the function ϕ on $\mathbb{T}^1 \times [0, 2\pi]$, where ϕ is such that $\gamma_{(x,0)}(s) \doteq (\phi(x,s), s)$ is the integral curve of $\Re\mathsf{L}$ with initial point $(x, 0)$. It is easy to see that there is an open set $\mathcal{U}' \subset \mathcal{U}$ such that for every $(x, 0) \in \mathcal{U}' \times \mathbb{T}^1$ one has $(\phi(x,s), s) \in \mathcal{U} \times \mathbb{T}^1$ if $0 \leqslant s \leqslant 2\pi$. It follows that, for every $(x,0) \in \mathcal{U}' \times \mathbb{T}^1$ and $0 \leqslant s \leqslant 2\pi$, we have:

$$\left|\phi(x,s) - x\right| \leqslant \frac{1}{8}|x - x_0|; \tag{2.3}$$

indeed, for some s', $|\phi(x,s) - x| = |\phi(x,s) - \phi(x,0)| \leqslant |a(\phi(x,s'),s')|2\pi \leqslant 2\pi\frac{1}{16\pi C'}|\phi(x,s') - x_0| = \frac{1}{8C'}|\phi(x,s') - \phi(x_0,s')| \leqslant \frac{1}{8C'}C'|x - x_0| = \frac{1}{8}|x - x_0|$. Hence if $x \in \mathcal{U}'$, $x > x_0$ and $0 \leqslant s \leqslant 2\pi$, then $\phi(x,s) = \phi(x,s) - x + x \leqslant \frac{x-x_0}{8} + x = \frac{9x-x_0}{8}$ and, therefore,

$$x_0 < \phi(x,s) \leqslant \frac{9x - x_0}{8}. \tag{2.4}$$

Taking a subsequence (if necessary), we may assume that $(x_n, 0) \in \mathcal{U}' \times \mathbb{T}^1$, for all n.

Define $\tilde{\varphi}_n(x,t) = \varphi_n(x) \otimes 1_t$. Given $N \in \mathbb{N}$, a simple computation yields positive constants C_1 and C_2, depending on N, such that the following inequalities are satisfied:

$$\left|\varphi_n^{(j)}(x)\right| \leqslant C_1|x_n - x_0|^{-j}, \quad \text{for all } j \leqslant N; \tag{2.5}$$

$$\left|\partial^\alpha(a + \mathrm{i}b)(x,t)\right| \leqslant C_2|x - x_0|^{N+3}, \quad \text{for all } |\alpha| \leqslant N, \text{ and for all } t \in \mathbb{T}^1. \tag{2.6}$$

Thus, given $N \in \mathbb{N}$, inequalities (2.5) and (2.6) imply that there is a positive constant C_3 such that the following estimate holds:

$$\left\|(a + \mathrm{i}b)\tilde{\varphi}_n\right\|_{(N)} \leqslant C_3|x_n - x_0|^3, \quad n \in \mathbb{Z}_+. \tag{2.7}$$

An important fact which will be used several times in the sequel is that, thanks to condition $(\mathcal{P})$ we have:

$$\left|\int_{r_1}^{r_2} b \circ \gamma_{(x,0)}(s)\,\mathrm{d}s\right| = \int_{r_1}^{r_2} \left|b \circ \gamma_{(x,0)}(s)\right|\mathrm{d}s, \tag{2.8}$$

for all $r_1 < r_2 < \infty$ and for every integral curve $\gamma_{(x,0)}$ of $\Re\mathsf{L}$.

As mentioned before, we will assume that the relevant curves in each case of Lemma 2.3 lie in the region $x > x_0$.

Suppose that case $(\star.1)$ occurs. Then we can find a sequence $((x_n, 0))$, with (x_n) strictly decreasing to x_0, such that through each point $(x_n, 0)$ there passes a periodic integral curve of $\Re\mathsf{L}$, namely, $\gamma_n(s) \doteq \gamma_{(x_n,0)}(s)$, $s \in [0, 2\pi]$, and such that, furthermore, $b \circ \gamma_n \not\equiv 0$. Consider the sequence of functions (φ_n) given by (2.2), corresponding to the our sequence (x_n) above; write $\tilde{\varphi}_n = \varphi_n \otimes 1_t$.

Define

$$\mu_n(\varphi) \doteq \alpha_n \int_0^{2\pi} \varphi \circ \gamma_n(s)\,\mathrm{d}s, \quad \forall \varphi \in C^\infty(\mathbb{T}^2),$$

where $\alpha_n = \frac{1-\cos(x_n - x_0)}{\int_0^{2\pi} |b\circ\gamma_n|(s)\,\mathrm{d}s}$. Clearly $\mu_n \in \mathcal{D}'(\mathbb{T}^2)$. Also, if $\varphi \in C^\infty(\mathbb{T}^2)$, then we have:

$$\begin{aligned} {}^t\mathsf{L}\mu_n(\varphi) &= \alpha_n\left(\int_0^{2\pi} (\partial_t\varphi + a\partial_x\varphi)\circ\gamma_n(s)\,\mathrm{d}s + \mathrm{i}\int_0^{2\pi} (b\partial_x\varphi)\circ\gamma_n(s)\,\mathrm{d}s\right) \\ &= \alpha_n\left(\int_0^{2\pi} (\varphi\circ\gamma_n)'(s)\,\mathrm{d}s + \mathrm{i}\int_0^{2\pi} (b\partial_x\varphi)\circ\gamma_n(s)\,\mathrm{d}s\right). \end{aligned}$$

Note that $\int_0^{2\pi} (\varphi\circ\gamma_n)'(s)\,\mathrm{d}s = 0$. Hence we have

$$\left|{}^t\mathsf{L}\mu_n(\varphi)\right| \leqslant \alpha_n \sup_{\mathbb{T}^2} |\partial_x\varphi| \int_0^{2\pi} |b\circ\gamma_n(s)|\,\mathrm{d}s = \bigl(1-\cos(x_n - x_0)\bigr)\sup_{\mathbb{T}^2}|\partial_x\varphi|.$$

Therefore ${}^t\mathsf{L}\mu_n \to 0$ weakly. It follows from (2.4) and from the fact that $\varphi_n \equiv 1$ on $[x_0, (9x_n - x_0)/8]$ that $\tilde{\varphi}_n \circ \gamma_n \equiv 1$; this together with (2.8) and inequalities (2.1) and (2.7) imply that

$$\begin{aligned} 1-\cos(x_n - x_0) &= \frac{1-\cos(x_n - x_0)}{\int_0^{2\pi} |b\circ\gamma_n(s)|\,\mathrm{d}s}\left|\int_0^{2\pi} (b\tilde{\varphi}_n)\circ\gamma_n(s)\,\mathrm{d}s\right| = \alpha_n\left|\int_0^{2\pi} (b\tilde{\varphi}_n)\circ\gamma_n(s)\,\mathrm{d}s\right| \\ &\leqslant \bigl|\mu_n\bigl[(a+\mathrm{i}b)\tilde{\varphi}_n\bigr]\bigr| \leqslant CC_3|x_n - x_0|^3, \end{aligned}$$

a contradiction, which concludes the proof in case ($\star$.1).

Suppose now that case ($\star$.2) occurs. Then, there exist sequences (x_n), (x'_n) and (x''_n), each of which is strictly decreasing and converges to x_0, such that $x_n \in (x''_n, x'_n)$ and, furthermore, the integral curves, $\gamma_{(x''_n,0)}$ and $\gamma_{(x'_n,0)}$, of $\Re\mathsf{L}$, are periodic, the integral curve, $\gamma_n = \gamma_{(x_n,0)}$, through each $(x_n, 0)$, is a non-periodic integral curve of $\Re\mathsf{L}$, and we have: one of the curves $\gamma_{(x''_n,0)}$, $\gamma_{(x'_n,0)}$, is the α-limit set of $\gamma_{(x_n,0)}$, while the other curve is the ω-limit set of $\gamma_{(x_n,0)}$.

Hence we can find $s_n < s'_n \in \mathbb{R}$ such that $\gamma_n(s_n)$ and $\gamma_n(s'_n)$ belong to $\mathbb{T}^1 \times \{0\}$ and furthermore

$$\frac{n}{n+1} \leqslant \frac{|\phi(x_n, s'_n) - \phi(x_n, s_n)|}{x'_n - x''_n} < 1.$$

Consider the sequence (φ_n), given by (2.2), corresponding to the sequence (x'_n). We may assume (by taking a subsequence, if necessary) that one of the following situations occurs:

(i) $\int_{s_n}^{s'_n} |b\circ\gamma_n|(s)\,\mathrm{d}s \leqslant x'_n - x''_n$, for all n;
(ii) $\int_{s_n}^{s'_n} |b\circ\gamma_n|(s)\,\mathrm{d}s > x'_n - x''_n$, for all n.

Suppose first that case (i) occurs. Define

$$\mu_n(\varphi) \doteq \frac{1-\cos(x'_n - x_0)}{x'_n - x''_n}\int_{s_n}^{s'_n} \varphi\circ\gamma_n(s)\,\mathrm{d}s, \quad \forall\varphi \in C^\infty(\mathbb{T}^2).$$

Clearly $\mu_n \in \mathcal{D}'(\mathbb{T}^2)$. Also, if $\varphi \in C^\infty(\mathbb{T}^2)$ and K is a Lipschitz constant for φ, we have $|{}^t\mathsf{L}\mu_n(\varphi)| \leqslant [1-\cos(x'_n - x_0)](K + \sup_{\mathbb{T}^2}|\partial_x\varphi|)$ hence ${}^t\mathsf{L}\mu_n \to 0$ weakly. As in case ($\star$.1) we have, from (2.4), $\tilde{\varphi}_n \circ \gamma_n \equiv 1$, hence by using inequalities (2.1) and (2.7), we obtain:

$$\frac{n}{n+1}\big(1-\cos(x_n'-x_0)\big) \leqslant \big(1-\cos(x_n'-x_0)\big)\frac{|\phi(x_n,s_n')-\phi(x_n,s_n)|}{x_n'-x_n''}$$
$$= \frac{1-\cos(x_n'-x_0)}{x_n'-x_n''}\left|\int_{s_n}^{s_n'}(a\tilde{\varphi}_n)\circ\gamma_n(s)\right|\mathrm{d}s$$
$$\leqslant \big|\mu_n\big[(a+\mathrm{i}b)\tilde{\varphi}_n\big]\big| \leqslant CC_3|x_n'-x_0|^3.$$

Therefore

$$1-\cos(x_n'-x_0) \leqslant \frac{n+1}{n}CC_3|x_n'-x_0|^3 \leqslant 2CC_3|x_n'-x_0|^3,$$

which is a contradiction.

Suppose now that (ii) is satisfied. Then

$$\int_{s_n}^{s_n'}|b\circ\gamma_n|(s)\,\mathrm{d}s > x_n'-x_n'' > \big|\phi(x_n,s_n')-\phi(x_n,s_n)\big|.$$

Define:

$$\mu_n(\varphi) \doteq \frac{1-\cos(x_n'-x_0)}{\int_{s_n}^{s_n'}|b\circ\gamma_n|(s)\,\mathrm{d}s}\int_{s_n}^{s_n'}\varphi\circ\gamma_n(s)\,\mathrm{d}s, \quad \forall\varphi\in C^\infty\big(\mathbb{T}^2\big).$$

Clearly $\mu_n\in\mathcal{D}'(\mathbb{T}^2)$. Also, if $\varphi\in C^\infty(\mathbb{T}^2)$ and K is a Lipschitz constant for φ, then $|{}^t\mathsf{L}\mu_n(\varphi)| \leqslant [1-\cos(x_n'-x_0)](K+\sup_{\mathbb{T}^2}|\partial_x\varphi|)$, hence ${}^t\mathsf{L}\mu_n\to 0$ weakly. As before, (2.4) implies that $\tilde{\varphi}_n\circ\gamma_n\equiv 1$, hence (2.8) and inequalities (2.1) and (2.7) imply that

$$1-\cos(x_n'-x_0) = \frac{1-\cos(x_n'-x_0)}{\int_{s_n}^{s_n'}|b\circ\gamma_n(s)|\,\mathrm{d}s}\int_{s_n}^{s_n'}|(b\tilde{\varphi}_n)\circ\gamma_n(s)|\,\mathrm{d}s$$
$$\leqslant \big|\mu_n\big[(a+\mathrm{i}b)\tilde{\varphi}_n\big]\big| \leqslant CC_3|x_n-x_0|^3,$$

a contradiction, which concludes the proof in case ($\star$.2).

Finally, suppose that ($\star$.3) occurs. We may assume, without loss of generality, that for each $x>0$, the integral curve $\gamma_{(x,0)}$ has $\{x_0\}\times\mathbb{T}^1$ as its ω-limit set (in case it is the α-limit set the proof is analogous). We may assume that there is a sequence (x_n), strictly decreasing to x_0, such that the integral curve $\gamma_n(s)=\gamma_{(x_n,0)}(s)=(\phi(x_n,s),s)$, has $\{x_0\}\times\mathbb{T}^1$ as its ω-limit set. Consider the sequence of functions given by (2.2), corresponding to this sequence (x_n). For each n choose $s_n\in\mathbb{R}_+$ such that $(x_n',0)=\gamma_{(x_n,0)}(s_n)$ satisfies $|x_n'-x_0|<|x_n-x_0|^3$. We may assume (by taking a subsequence, if necessary) that one of the following possibilities occurs:

(i) $\int_0^{s_n}|b\circ\gamma_n|(s)\,\mathrm{d}s\to 0$;
(ii) $\int_0^{s_n}|b\circ\gamma_n|(s)\,\mathrm{d}s\neq 0$, $\forall n$, and $\int_0^{s_n}|b\circ\gamma_n|(s)\,\mathrm{d}s\to C$, $0<C\leqslant\infty$.

Suppose that item (i) is satisfied. Define:

$$\mu_n(\varphi)\doteq\int_0^{s_n}\varphi\circ\gamma_n(s)\,\mathrm{d}s, \quad \forall\varphi\in C^\infty\big(\mathbb{T}^2\big).$$

Clearly $\mu_n\in\mathcal{D}'(\mathbb{T}^2)$. Also, if $\varphi\in C^\infty(\mathbb{T}^2)$, we have $|{}^t\mathsf{L}\mu_n(\varphi)|\leqslant|\varphi(\gamma_n(s_n))-\varphi(\gamma_n(0))|+\sup_{\mathbb{T}^2}|\partial_x\varphi|\int_0^{s_n}|b\circ\gamma_n(s)|\,\mathrm{d}s$. Therefore ${}^t\mathsf{L}\mu_n\to 0$ weakly. Hence, from inequalities (2.1) and (2.7), we obtain $|\mu_n[(a+\mathrm{i}b)\tilde{\varphi}_n]|\leqslant CC_3|x_n-x_0|^3$. Also, since (2.4) implies that $\tilde{\varphi}_n\circ\gamma_{n|_{[0,\infty)}}\equiv 1$, we have:

$$\left|\mu_n\left[(a+\mathrm{i}b)\tilde{\varphi}_n\right]\right| \geqslant \left|\int_0^{s_n} (a\tilde{\varphi}_n)\circ\gamma_n(s)\,\mathrm{d}s\right| = \left|\int_0^{s_n} a\circ\gamma_n(s)\,\mathrm{d}s\right| = \left|\int_0^{s_n} \partial_s\phi(x_n,s)\,\mathrm{d}s\right|$$
$$= |x'_n - x_n| = x_n - x'_n = x_n - x_0 - (x'_n - x_0).$$

Hence

$$|x_n - x_0| \leqslant (CC_3+1)|x_n - x_0|^3,$$

which is a contradiction.

Suppose now that (ii) is satisfied. Define:

$$\mu_n(\varphi) \doteq \frac{1-\cos(x_n-x_0)}{\int_0^{s_n}|b\circ\gamma_n|(s)\,\mathrm{d}s}\int_0^{s_n}\varphi\circ\gamma_n(s)\,\mathrm{d}s, \quad \forall\varphi\in C^\infty(\mathbb{T}^2).$$

Clearly $\mu_n \in \mathcal{D}'(\mathbb{T}^2)$. Also, if $\varphi\in C^\infty(\mathbb{T}^2)$, we have:

$$\left|{}^t\mathsf{L}\mu_n(\varphi)\right| \leqslant \left(1-\cos(x_n-x_0)\right)\left(\frac{|\varphi(\gamma_n(s_n)) - \varphi(\gamma_n(0))|}{\int_0^{s_n}|b\circ\gamma_n|(s)\,\mathrm{d}s} + \sup_{\mathbb{T}^2}|\partial_x\varphi|\right),$$

hence ${}^t\mathsf{L}\mu_n \to 0$ weakly. As before, we obtain:

$$1-\cos(x_n-x_0) = \frac{1-\cos(x_n-x_0)}{\int_0^{s_n}|b\circ\gamma_n(s)|\,\mathrm{d}s}\left|\int_0^{s_n}(b\tilde{\varphi}_n)\circ\gamma_n(s)\,\mathrm{d}s\right|$$
$$\leqslant \left|\mu_n\left[(a+\mathrm{i}b)\tilde{\varphi}_n\right]\right| \leqslant CC_3|x_n-x_0|^3,$$

which is a contradiction. The proof is complete. □

Remark 2.5. The same arguments of the proof of the Theorem 2.4 can be applied to prove a necessary condition for the global solvability for all vector fields, of the form (1.1), which satisfy condition $(\mathcal{P})$ and have at least one Sussmann orbit which, although not equal to $\{x\}\times\mathbb{T}^1$, is homotopic to $\{0\}\times\mathbb{T}^1$ and is contained in $\partial\mathcal{K}$. Indeed, if γ is such orbit then by [8] (Theorem 2.1) and [11] (Theorem 1.3, Chapter 8) we obtain, as in [4], a diffeomorphism of $\mathbb{T}^2$ onto $\mathbb{T}^2$ such that in the new coordinates L can be written in the form:

$$\mathsf{L} = \partial/\partial t + \left(\tilde{a}(x,t) + \mathrm{i}\tilde{b}(x,t)\right)\partial/\partial x, \quad \tilde{a},\tilde{b}\in C^\infty(V;\mathbb{R}),$$

in a neighborhood, V, of $\{x_0\}\times\mathbb{T}^1$, and γ can be written in the form $\gamma(s) = (x_0, s)$, $s\in[0,2\pi]$.

The next result is a consequence of Theorem 2.4 (see also [5]).

Corollary 2.6. *If $\mathsf{L} = \partial/\partial t + (a(x)+\mathrm{i}b(x))\partial/\partial x$, with $b\not\equiv 0$, is globally solvable and satisfies condition $(\mathcal{P})$ then the zeros of $a+\mathrm{i}b$ are of finite order.*

To conclude this section we make just one remark: the global solvability of L does not imply, in general, that condition $(\mathcal{P})$ holds; this happens, for instance, in the case when $\mathsf{L} = \partial/\partial t + \mathrm{i}\sin t\,\partial/\partial x$ (see [15]).

3. Sufficient conditions for strong solvability

In this section, our goal is to present sufficient conditions for L, having the form (1.1), that is,

$$\mathsf{L} = \partial/\partial t + \left(a(x,t) + \mathrm{i}b(x,t)\right)\partial/\partial x, \quad a,b\in C^\infty\left(\mathbb{T}^2;\mathbb{R}\right),\ b\not\equiv 0,$$

to be strongly solvable. In particular, we will obtain sufficient conditions for L to be globally solvable.

3.1. The assumptions and preliminary results

We will assume that condition $(\mathcal{P})$ is satisfied; recall that condition $(\mathcal{P})$ is necessary for strong solvability, in view of Hörmander's results (see [14], Corollary 26.4.8).

We begin by analyzing several of the possibilities for the integral curves of $\Re\mathsf{L}$ and for the one-dimensional Sussmann orbits of the structure defined by L. We remark that, in our case, a one-dimensional Sussmann orbit is an integral curve of $\Re\mathsf{L}$ along which $b\equiv 0$, since the coefficient of $\partial/\partial t$ in $\Re\mathsf{L}$ is equal to 1 everywhere.

Let us consider the case where each integral curve of $\Re\mathsf{L}$ is periodic. Then by using a unimodular transformation, if necessary, we may assume that each integral curve is homotopic to $\{0\}\times\mathbb{T}^1$, and then the flow of $\Re\mathsf{L}$ gives rise to a diffeomorphism of $\mathbb{T}^2$ which transforms our vector field into a new vector field whose real part is equal to $\partial/\partial t$, a case which is already treated in [1].

Therefore we have the right to restrict ourselves to the case where $\Re\mathsf{L}$ has at least one non-periodic integral curve.

Let us now consider the case where L has no periodic, one-dimensional Sussmann orbits, that is, there is no periodic integral curve of $\Re\mathsf{L}$ along which $b\equiv 0$. Then there are two possible cases. First case: there are no periodic integral curves of $\Re\mathsf{L}$. Second case: $b\not\equiv 0$ along each periodic integral curve of $\Re\mathsf{L}$.

Before analyzing these two cases, we recall the fact that, as an immediate consequence of [22] (see also [16]), we have: the α-limit (or the ω-limit) set of any integral curve of the real vector field $\partial/\partial t + a(x,t)\partial/\partial x$ on $\mathbb{T}^2$ is either a periodic integral curve or $\mathbb{T}^2$.

First case: there are no periodic integral curves of $\Re\mathsf{L}$. Then the above-mentioned result of [22] implies that each integral curve of $\Re\mathsf{L}$ is dense in $\mathbb{T}^2$; since $b\not\equiv 0$, we conclude that $\mathbb{T}^2$ is the only Sussmann orbit. Let γ_p be a bicharacteristic curve of $\Re\ell$ through $p\in\mathcal{C}$, where $\mathcal{C}$ is the characteristic set, and $\ell=\tau+[a(x,t)+\mathrm{i}b(x,t)]\xi$ is the principal symbol of L. If $\gamma_p(s)\in\mathcal{C}$, for all $s\in D_{\gamma_p}$, then the projection of γ_p on $\mathbb{T}^2$ is an one-dimensional orbit; hence we have that condition (i) of Theorem 7.1 of [12] is satisfied. Therefore L is strongly solvable.

Second case: $b\not\equiv 0$ along each periodic integral curve of $\Re\mathsf{L}$. Then a simple argument shows that again $\mathbb{T}^2$ is the only Sussmann orbit and L is strongly solvable.

Therefore we have the right to restrict ourselves to the case where L has at least one periodic, one-dimensional Sussmann orbit; in particular, $\Re\mathsf{L}$ has at least one periodic integral curve.

Recall that $\mathcal{K}$ denotes the union of all one-dimensional, *periodic* Sussmann orbits of the structure defined by L. If there were an infinite number of such periodic one-dimensional orbits then, by using delta distributions concentrated on each orbit, we would prove that $\dim(\ker{}^t\mathsf{L})=\infty$, hence L would not be strongly solvable.

Summing up the above discussion, in order to study the strong solvability of L, as in (1.1), we have the right to restrict ourselves to the case where L satisfies the following conditions:

- there is at least one periodic, one-dimensional Sussmann orbit (without loss of generality, we will assume further that each such orbit is homotopic to $\{0\}\times\mathbb{T}^1$);
- $\Re\mathsf{L}$ has at least one non-periodic integral curve;
- the number of periodic, one-dimensional orbits is finite.

So, in this section we will assume that the above conditions are satisfied. In particular, we will suppose that $\mathcal{K}=\bigcup_{k=1}^{q}\gamma_k$, for some q. Define $\mathcal{U}=\mathbb{T}^2\setminus\mathcal{K}$.

For convenience, set $\gamma_{q+1}=\gamma_1$. This enumeration of the curves in $\mathcal{K}$ is such that there is no γ_ℓ between γ_k and γ_{k+1}, for $k=1,\dots,q$. Thus the open set bounded by γ_k and γ_{k+1} is a connected component of $\mathcal{U}$.

Lemma 3.1. *Suppose that* L *satisfies condition* $(\mathcal{P})$. *Let* V *be a two-dimensional orbit of the structure defined by* L *on* $\mathbb{T}^2$. *Let* $s\in\mathbb{R}$. *If* $\mu\in\mathcal{D}'(V)$ *and* ${}^t\mathsf{L}\mu\in H^s_{\mathrm{loc}}$ *then* $\mu\in H^s_{\mathrm{loc}}$.

Proof. The proof is an adaptation of that of a similar result in [1] (Lemma 3.1) (there, it is assumed that $\Re\mathsf{L}=\partial/\partial t$). Let $(x_0,t_0)\in V$. Then the integral curve, γ, of $\Re\mathsf{L}$ such that $\gamma(0)=(x_0,t_0)$, is contained in V which is a two-dimensional orbit. In particular, γ is not a one-dimensional orbit, hence there is s_0 such that $b(\gamma(s_0))\neq 0$. Thus L is elliptic at $\gamma(s_0)$ hence $\mu\in H^{s+1}\subset H^s$ at $\gamma(s_0)$. In view of condition $(\mathcal{P})$, b has constant sign in a neighborhood of the arc $\gamma([0,s_0])$, hence Theorem 3.1 in [12] implies that $\mu\in H^s$ at (x_0,t_0). $\square$

Lemma 3.2. *Let V be a two-dimensional orbit of the structure defined by* L *on* $\mathbb{T}^2$. *Let* $\mu \in \mathcal{D}'(V)$ *be such that* ${}^t\mathsf{L}\mu = 0$. *If* $\mu = 0$ *in some open subset of* V *then* $\mu \equiv 0$.

Proof. The support of μ is equal to a union of Sussmann orbits ([18], Proposition 2.3). Hence $\mu \equiv 0$. $\square$

Since $\mathcal{K}$ is finite union of periodic, one-dimensional orbits, we can find—with the help of results in [8,11]—a diffeomorphism from $\mathbb{T}^2$ into $\mathbb{T}^2$ such that, using the same notation (x,t) for the new coordinates, L has the same form as originally, and $\mathcal{K}$ can be written as $\mathcal{K} = \mathcal{N} \times \mathbb{T}^1$, where

$$\mathcal{N} = \{x \in \mathbb{T}^1; a(x,t) + \mathrm{i}b(x,t) = 0, \forall t \in \mathbb{T}^1\} = \{x_1, \dots, x_q\}, \quad \text{and}$$

$$\mathcal{U} = \mathbb{T}^2 \setminus (\mathcal{N} \times \mathbb{T}^1) = \bigcup_{k=1}^{q} J_k \times \mathbb{T}^1, \quad \text{where } J_k = (x_k, x_{k+1}).$$

For each $x_k \in \mathcal{N}$, define:

$$\mathcal{F}(x_k) = \ker {}^t\mathsf{L} \cap \mathcal{E}'(\{x_k\} \times \mathbb{T}^1).$$

If $J_{k-1} \times \mathbb{T}^1$ and $J_k \times \mathbb{T}^1$ are two-dimensional orbits and $U = (x_k - \delta, x_k + \delta) \times \mathbb{T}^1$, with $\delta > 0$ small, then Lemma 3.2 implies that $\ker {}^t\mathsf{L} \cap \mathcal{E}'(\overline{U}) = \mathcal{F}(x_k)$.

We also use the notation $f \in \mathcal{F}(x_k)^\circ$ to mean that $\langle h, f\rangle = 0$ for every $h \in \mathcal{F}(x_k)$.

Lemmas 3.1 and 3.2 allow us to easily adapt the (non-trivial) proof of Theorem 3.1 in [1], thus yielding the following extension:

Theorem 3.3. *Suppose that* L *satisfies condition* $(\mathcal{P})$, *that* $\mathcal{K} = \mathcal{N} \times \mathbb{T}^1$, *with* $\mathcal{N}$ *finite, and that each connected component of* $\mathcal{U}$ *is a two-dimensional orbit. Then* L *is strongly solvable if and only if, for each* $x_k \in \mathcal{N}$, *the following properties are satisfied*:

($\star$) $\dim \mathcal{F}(x_k) < \infty$;
($\star\star$) *for each* f *smooth near* $\{x_k\} \times \mathbb{T}^1$, *with* $f \in \mathcal{F}(x_k)^\circ$, *there exists a smooth solution,* u, *to the equation* $\mathsf{L}u = f$ *in a neighborhood of* $\{x_k\} \times \mathbb{T}^1$.

The usefulness of this result lies in the fact that, in order to study the strong solvability on the torus, one is reduced to studying solvability near the characteristic set (the latter type of solvability was the subject of several recent works, such as [2,3]); see also [7,20] for a study of equations near the characteristic set, as in property ($\star\star$) above.

3.2. Statement of the main result

We are now ready to state the main result of this section.

Theorem 3.4. *Consider the vector field,*

$$\mathsf{L} = \partial/\partial t + (a(x,t) + \mathrm{i}b(x,t))\partial/\partial x, \quad a, b \in C^\infty(\mathbb{T}^2; \mathbb{R}), \; b \not\equiv 0,$$

on the two-torus $\mathbb{T}^2$, *and assume that* L *satisfies condition* $(\mathcal{P})$. *Suppose that* L *has a finite number of periodic, one-dimensional orbits, namely,* $\{x_k\} \times \mathbb{T}^1$, $x_k \in \mathcal{N}$, *and call* $\mathcal{K}$ *the union of such orbits. Assume that each connected component of* $\mathbb{T}^2 \setminus \mathcal{K}$ *is a two-dimensional orbit. Suppose that, for each* k, *there exists* $m_k \geqslant 2$ *such that, in a neighborhood of* $\{x_k\} \times \mathbb{T}^1$, *we have* $a(x,t) + \mathrm{i}b(x,t) = (x - x_k)^{m_k}(a_0(x,t) + \mathrm{i}b_0(x,t))$, *where* a_0, b_0 *are smooth and* $t \mapsto b_0(x,t) \not\equiv 0$. *Then* L *is strongly solvable.*

Remark 3.5. If one does not assume that the number of one-dimensional orbits is finite then L is not strongly solvable, as explained in the previous subsection.

Remark 3.6. Since $\mathcal{K}$ is a finite union we may assume that $\mathcal{K} = \mathcal{N} \times \mathbb{T}^1$, with $\mathcal{N}$ finite as explained after Lemma 3.2.

Remark 3.7. If one does not assume that $a + \mathrm{i}b$ vanishes to finite order along each $\{x_k\} \times \mathbb{T}^1$ then L is not even globally solvable, in view of Theorem 2.4.

Remark 3.8. If one does not assume that each connected component of $\mathcal{U}$ is a two-dimensional orbit, then L may not be strongly solvable. For instance, if $\mathsf{L} = \partial/\partial t + (a(x) + \mathrm{i}b(x))\partial/\partial x$ and if $b \equiv 0$ over some J_k, then $\dim(\ker {}^t\mathsf{L}) = \infty$ and L is not strongly solvable (see [5]).

Remark 3.9. In the work [5] the special class of vector fields $\mathsf{L} = \partial/\partial t + (a(x) + \mathrm{i}b(x))\partial/\partial x$ was studied; note that in this case a and b depend only on the variable x. Suppose that $(a + \mathrm{i}b)(x) = x^{n_k} a_0(x) + \mathrm{i}x^{m_k} b_0(x)$, in some neighborhood of x_k. It was shown there that if, for instance, $a(x_k) \neq 0$ and $m_k > 2n_k - 1$, for some k, then L is not strongly solvable. Our assumption in the Theorem 3.4 is equivalent to $2 \leqslant m_k \leqslant n_k$ and $t \mapsto b_0(x,t) \not\equiv 0$.

We will now move on to the proof of Theorem 3.4 and for that we will prove that properties ($\star$) and ($\star\star$) of Theorem 3.3 are satisfied.

3.3. Beginning of the proof of Theorem 3.4: solving modulo flat functions

Fix $x_k \in \mathcal{N}$. We may assume, without loss of generality, that $x_k = 0$.

Lemma 3.10. *Assume that condition ($\mathcal{P}$) holds. Let $c(x,t) \sim c_m(t)x^m + c_{m+1}(t)x^{m+1} + \cdots$ be the Taylor expansion of $c = a + \mathrm{i}b$ at $\{0\} \times \mathbb{T}^1$, where $m \geqslant 1$ and $b_m(t) \not\equiv 0$. Then,*

(i) *b_m does not change sign*;
(ii) $\Im(\int_0^{2\pi} c_m(t)\,\mathrm{d}t) = \int_0^{2\pi} b_m(t)\,\mathrm{d}t \neq 0$.

Proof. The proof is an adaptation of that of a similar result in [1] (Lemma 4.1) (there, it is assumed that $a \equiv 0$). It suffices to use the fact that b does not change sign over each two-dimensional orbit ([17], Theorem 3.1). □

Now if in the proof of Proposition 4.1 in [1] we substitute $c_m(t)$ for $\mathrm{i}b_m(t)$ and then use lemma above, we obtain the following result:

Lemma 3.11. *Under the assumptions of lemma above, suppose that $m \geqslant 2$. Then,*

(i) *The equation $\mathsf{L}u = f$ can be solved modulo flat functions at $\{0\} \times \mathbb{T}^1$, provided that $(\delta^{(j)}(x) \otimes 1_t)(f) = 0$ for all $0 \leqslant j \leqslant m-1$;*
(ii) *$\mathcal{F}(0)$ is spanned by distributions $\delta^{(j)}(x) \otimes 1_t$, $0 \leqslant j \leqslant m-1$.*

It follows from the above proposition that in order to prove Theorem 3.4 it suffices to solve the equation $\mathsf{L}u = f$, when f is flat at $\{0\} \times \mathbb{T}^1$.

3.4. Method of descent

Next, as in [1], we will make use of the so-called method of descent (see also [10,9]), which we now recall.

We must solve

$$\mathsf{L}u = f \tag{3.1}$$

in $\Omega \doteq (-\delta, \delta) \times \mathbb{T}^1$, where $f \in C^\infty(\Omega)$ vanishes to infinite order along $\{0\} \times \mathbb{T}^1$. The assumptions of Theorem 3.4 allow us to assume (by contracting δ if necessary) that $c(x,t) = x^m(a_0(x,t) + \mathrm{i}b_0(x,t))$ in Ω, where $m \geqslant 2$, and the function $x \mapsto b_0(x, t_0) \neq 0$, $\forall x \in (-\delta, \delta)$, for some t_0, which, without loss of generality, we may take to be 0. Since L satisfies condition ($\mathcal{P}$), we may assume, without loss of generality that, $b_0 \geqslant 0$. Thus we may write L, in Ω, in the form:

$$\mathsf{L} = \partial/\partial t + x^m c_0(x,t)\partial/\partial x, \tag{3.2}$$

where $c_0(x,t)=a_0(x,t)+\mathrm{i}b_0(x,t)$, with $b_0(x,t)\geqslant 0$, for any (x,t), and $b_0(x,0)>0$, for every x. We are now ready to apply the method of descent: we add the variable y and consider the following operator in $(-\delta,\delta)\times\mathbb{R}\times\mathbb{T}^1$:

$$\mathcal{L}=\partial/\partial t+c_0(x,t)\big(x^m\partial/\partial x+\partial/\partial y\big). \tag{3.3}$$

The solvability of (3.1) is equivalent to the solvability of the overdetermined system

$$\begin{cases}\mathcal{L}u=f,\\ \partial u/\partial y=0.\end{cases} \tag{3.4}$$

Define new coordinates: $\bar t=t$, $\bar y=y$, and $\bar x=x(1+(m-1)yx^{m-1})^{-1/(m-1)}$. We have $(x^m\partial/\partial x+\partial/\partial y)\bar x=0$, $\partial/\partial y\bar x=-\bar x^m$, $\partial/\partial\bar y=x^m\partial/\partial x+\partial/\partial y$, and $\partial/\partial y=\partial/\partial\bar y-\bar x^m\partial/\partial\bar x$.

The operator $\mathcal{L}$, in the coordinates $(\bar x,\bar y,\bar t)$ is:

$$\mathcal{L}^\sharp=\partial/\partial\bar t+C(\bar x,\bar y,\bar t)\partial/\partial\bar y, \tag{3.5}$$

where $C(\bar x,\bar y,\bar t)=c_0(x(\bar x,\bar y),\bar t)$. Our next goal will then be to solve the system corresponding to (3.4), namely,

$$\begin{cases}\mathcal{L}^\sharp u^\sharp=f^\sharp,\\ \mathcal{X}^\sharp u^\sharp=0,\end{cases} \tag{3.6}$$

where

$$\mathcal{X}^\sharp=\partial/\partial\bar y-\bar x^m\partial/\partial\bar x. \tag{3.7}$$

We remark that $x=\bar x\mu(\bar x,\bar y)$, for some non-vanishing μ. This implies that the function $f^\sharp(\bar x,\bar y,\bar t)\doteq f(\bar x\mu(\bar x,\bar y),\bar t)$ is flat at $\bar x=0$, and furthermore satisfies the compatibility condition

$$\mathcal{X}^\sharp f^\sharp=0. \tag{3.8}$$

Finally note that $[\mathcal{L}^\sharp,\mathcal{X}^\sharp]=0$.

3.5. The new system

In this section, we will make a transformation of our system (3.6). For simplicity of notation, we will rename our variables and operators. In the new variables, still denoted (x,y,t), we will consider the operator:

$$\mathcal{L}=\partial/\partial t+c(x,y,t)\partial/\partial y,\quad \text{where } c=a+\mathrm{i}b, \tag{3.9}$$

defined on an open subset of $\mathbb{R}\times\mathbb{R}\times\mathbb{T}^1$ containing $\{0\}\times\mathbb{R}\times\mathbb{T}^1$, and we will assume that $b\geqslant 0$ and $b(0,y,0)\neq 0$, $\forall y\in\mathbb{R}$. We will also consider the operator:

$$\mathsf{X}\doteq\partial/\partial y-x^m\partial/\partial x, \tag{3.10}$$

where $m\geqslant 2$, and we will assume that $[\mathcal{L},\mathsf{X}]=0$, or equivalently that $\mathsf{X}c=0$ (for $[\mathcal{L},\mathsf{X}]=-(\mathsf{X}c)\partial/\partial y$). When $x=0$ we obtain $0=\mathsf{X}c=\partial c/\partial y$; therefore $c(0,y,t)$ does not depend on y, hence we have:

$$c(0,y,t)=a(0,y,t)+\mathrm{i}b(0,y,t)\doteq a_\circ(t)+\mathrm{i}b_\circ(t)\doteq c_\circ(t). \tag{3.11}$$

We may assume, without loss of generality (use the change of variable $y\mapsto\tau y$, if necessary) that

$$\int_0^{2\pi} b_\circ(t)\,\mathrm{d}t=2\pi. \tag{3.12}$$

Let $M,\lambda\in\mathbb{R}$ be such that

$$\lambda=\int_0^{2\pi} a_\circ(t)\,\mathrm{d}t \tag{3.13}$$

and

$$M=\sup_{0\leqslant t\leqslant 2\pi}\left|\int_0^{t} a_\circ(t')\,\mathrm{d}t'\right|. \tag{3.14}$$

It is easy to see that $\gamma_{(0,y,0)}(s) = (0, y + \int_0^s a_\circ(t)\,dt, s)$ is an integral curve of $\Re\mathcal{L}$ with initial point $(0, y, 0)$. Note that $\gamma_{(0,y,0)}(s) = T_y \circ \gamma_{(0,0,0)}(s)$, where $T_y : \{0\} \times \mathbb{R} \times \mathbb{T}^1 \to \{0\} \times \mathbb{R} \times \mathbb{T}^1$ is such that $T_y(0, y', t) = (0, y + y', t)$. Hence in order for us to have $\{0\} \times \{0\} \times \mathbb{T}^1 \subset \bigcup_{y\in[-\beta,\beta]} \gamma_{(0,y,0)}[0, 2\pi]$, $\beta > 0$, it is necessary that $\beta \geqslant M$.

Take positive real numbers δ and ϵ so that $\mathcal{L}$ and X are defined in $(-\delta, \delta) \times (-3M - 1, 3M + 1) \times \mathbb{T}^1$ and $b(x, y, t) \neq 0$ if $(x, y, t) \in (-\delta, \delta) \times (-3M - 1, 3M + 1) \times (-\epsilon, \epsilon)$.

Let the function $f \in C^\infty((-\delta, \delta) \times (-3M - 1, 3M + 1) \times \mathbb{T}^1)$ be flat at $x = 0$ and satisfying $\mathsf{X} f = 0$. Our goal is to solve the overdetermined system:

$$\begin{cases} \mathcal{L}u = f, \\ \mathsf{X}u = 0, \end{cases} \tag{3.15}$$

in some neighborhood of $\{0\} \times \{0\} \times \mathbb{T}^1$.

In the remainder of this section, K will denote a compact subset of $(-\delta, \delta) \times (-3M - 1, 3M + 1) \times \mathbb{T}^1$, containing $\{0\} \times \{0\} \times \mathbb{T}^1$.

Lemma 3.12. *The operator*

$$\mathcal{L} : C^\infty(K) \longrightarrow C^\infty(K)$$

is surjective.

Proof. We begin by recalling the notation of Theorem 7.3 in [12]. The characteristic set, $\mathcal{C}$, of $\mathcal{L}$ is the set of zeros of

$$\ell(x, y, t, \xi, \eta, \tau) = \tau + \big(a(x, y, t) + ib(x, y, t)\big)\eta, \quad \text{with } (\xi, \eta, \tau) \neq (0, 0, 0).$$

As in [12], let $\mathcal{C}_2$ be the subset of $\mathcal{C}$ where $d(\tau + a\eta)$ and $d(b\eta)$ are linearly independent and let $\mathcal{C}_2^e$ be the union of all semi-bicharacteristic curves which intersect $\mathcal{C}_2$. Let $\mathcal{C}_{11}$ be the set of all points in $\mathcal{C} \setminus \mathcal{C}_2^e$ which can be connected to a non-characteristic point by means of a semi-bicharacteristic curve. From [12] (Proposition 2.1) it follows that $\mathcal{C}_2^e \cup \mathcal{C}_{11} \subset \mathcal{C}$. Since,

$$H_{\Re(\ell)} = \partial_t + a(x, y, t)\partial_y - \eta\big[a_x(x, y, t)\partial_\xi + a_y(x, y, t)\partial_\eta + a_t(x, y, t)\partial_\tau\big],$$

given $p_0 = (x_0, y_0, t_0, \xi_0, \eta_0, \tau_0) \in \mathcal{C}$, it is easy to see that a bicharacteristic curve of $\Re\ell$ with initial point p_0 has the form $\gamma_{p_0}(s) = (x_0, y(s), t_0 + s, \xi(s), \eta(s), \tau(s))$. If $p_1 = (x_0, y(s_0), t_0 + s_0, \xi(s_0), \eta(s_0), \tau(s_0))$, with $b(x_0, y(s_0), t_0 + s_0) \neq 0$, then it follows that:

(1) If $\eta(s_0) = 0$ then $\tau(s_0) = 0$, for $\tau(s) = -a(x_0, y(s), t_0 + s) \cdot \eta(s)$. Thus $\xi(s_0) \neq 0$ and, $d(\tau + a\eta)_{|p_1} = d\tau + a(x_0, y(s_0), t_0 + s_0)\,d\eta$ and $d(b\eta)_{|p_1} = b(x_0, y(s_0), t_0 + s_0)\,d\eta$. Hence $p_0 \in \mathcal{C}_2^e$, for $p_1 \in \mathcal{C}_2$.

(2) If $\eta(s_0) \neq 0$ then $p_0 \in \mathcal{C}_{11}$, for $p_1 \notin \mathcal{C}$.

Therefore, the set $\{p \in \mathcal{C};\ \exists s_0 \in I_{\gamma_p} \text{ with } b(x, y(s_0), t + s_0) \neq 0\}$ is contained in $\mathcal{C}_2^e \cup \mathcal{C}_{11}$. Thus, if $p_0 \in \mathcal{C} \setminus \mathcal{C}_2^e \cup \mathcal{C}_{11}$ then $b(x_0, y(s), t_0 + s) = 0$, $\forall s$, hence the projection of γ_{p_0}, into $\{x_0\} \times (-3M - 1, 3M + 1) \times \mathbb{T}^1$, is a one-dimensional orbit; recall that one-dimensional orbits of the structure defined by $\mathcal{L}$ are the integral curves, $s \mapsto (x, y(s), t + s)$, of $\Re\mathcal{L}$ such that $b(x, y(s), t + s) = 0$, for all s. Since there is $\epsilon > 0$ such that $b(x, y, t) \neq 0$ if $(x, y, t) \in V = (-\delta, \delta) \times (-3M - 1, 3M + 1) \times (-\epsilon, \epsilon)$, we have that each one-dimensional orbit over $\{x_0\} \times (-3M - 1, 3M + 1) \times \mathbb{T}^1$ tends to the boundary of such a set, as $s \to \omega_+(p_0)$ or $s \to \omega_-(p_0)$, because the curve $(x_0, y(s), t + s)$ is transversal to the direction y and does not intersect V (here, as in the theory of ordinary differential equations, $\omega_-(p_0)$ and $\omega_+(p_0)$ are, respectively, the left and the right end of the maximal interval, $I_{p_0} = (\omega_-(p_0), \omega_+(p_0))$, of definition of the curve γ_{p_0}). Therefore there is no one-dimensional orbit contained in K hence, if $p_0 \in \mathcal{C} \setminus \mathcal{C}_2^e \cup \mathcal{C}_{11}$, then the curve γ_{p_0} does not lie over K. Thus, in order to verify that $\mathcal{L}$ satisfies the assumptions of Theorem 7.3 in [12] it suffices to verify that no two-dimensional bicharacteristic lies over K. Let $\mathcal{B}$ be a two-dimensional bicharacteristic. If $p_0 \in \mathcal{B}$ then $\mathcal{B}$ contains the Sussmann orbit, $\mathcal{M}$, through p_0, of the set of Hamiltonians $\{H_{\Re\ell}, H_{\Im\ell}\}$, for they are tangent to $\mathcal{B}$. Since the integral curves of $H_{\Re\ell}$ and $H_{\Im\ell}$ project onto the integral curves of $\Re\mathcal{L}$ and $\Im\mathcal{L}$, respectively, one sees that the projection of $\mathcal{M}$ is indeed equal either to a one-dimensional or a two-dimensional orbit, $\mathcal{W}$, of the structure defined by $\mathcal{L}$. As before, if $\mathcal{W}$ is one-dimensional then $\mathcal{W}$ does not intersect V hence $\mathcal{M}$ does not lie over K. If now $\mathcal{W}$ is two-dimensional, then the previous reasoning says that we must assume, also, that $\mathcal{W}$ intersects V. Thus $\mathcal{W}$ contains V hence $\mathcal{M}$ does not lie over K. Therefore, from what we saw above, $\mathcal{B}$ does not lie over K.

Consider now the kernel:

$$\left\{v \in \mathcal{C}_c^\infty(K)\colon {}^t\mathcal{L}v = 0\right\}.$$

Since the support of v (v belonging to the set above) is a union of Sussmann orbits ([18], Proposition 2.3) and no such orbit is contained in K, we have:

$$\left\{v \in \mathcal{C}_c^\infty(K)\colon {}^t\mathcal{L}v = 0\right\} = \{0\}.$$

Therefore the final conclusion follows from [12] (Theorem 7.3). □

Lemma 3.13. *Let*

$$\mathcal{L}_0 = \partial/\partial t + c(0, y, t)\partial/\partial y,$$

and let $v_0 \in C^\infty(\{(y,t)\colon (0,y,t) \in K\})$ such that $\mathcal{L}_0 v_0 = 0$. Then there is $v \in C^\infty(K)$ satisfying $\mathcal{L}v = 0$ and $v(0,y,t) = v_0(y,t)$.

Proof. The argument (which we will recall now) is the one used in [12] (Corollary 4.4) and also in [1] (Lemma 5.1.2). Since $\mathcal{L}v_0(y,t)_{|x=0} = 0$, we have $\mathcal{L}v_0/x \in C^\infty(K)$. Now Lemma 3.12 implies the existence of $\tilde{v} \in \mathcal{C}^\infty(K)$ such that $\mathcal{L}\tilde{v} = \mathcal{L}v_0/x$. Define $v \doteq v_0 - x\tilde{v}$. Then $\mathcal{L}v = \mathcal{L}v_0 - x\mathcal{L}\tilde{v} = 0$ and $v(0,y,t) = v_0(y,t) - 0.\tilde{v}(0,y,t) = v_0(y,t)$. □

Proposition 3.14. *The following statements hold*:

(i) *Suppose that $f \in C^\infty(K)$ vanishes of order $m \geqslant 1$ at $x = 0$. Then there is a solution $v \in C^\infty(K)$ to the equation $\mathcal{L}v = f$, with v vanishing of order m at $x = 0$.*
(ii) *Let $f \in C^\infty(K)$ be flat at $x = 0$. Then there is a solution $v \in C^\infty(K)$ to the equation $\mathcal{L}v = f$, with v flat at $x = 0$.*

Proof. We will only prove (ii). The argument is the same as in [1] (Proposition 5.1.1). Lemma 3.12 implies the existence of a solution, $w \in C^\infty(K)$, to the equation $\mathcal{L}w = f$. We claim that there is a sequence of solutions, $\{h_j\}$, to the homogeneous equation $\mathcal{L}h_j = 0$, with $h_j \in C^\infty(K)$, such that

$$w_k(x,y,t) \doteq w(x,y,t) - \sum_{j=0}^{k} h_j(x,y,t)\frac{x^j}{j!} = \mathrm{O}\big(|x|^{k+1}\big), \tag{3.16}$$

for each k. Indeed, if $k = 0$, then we have $\mathcal{L}_0\{w(0,y,t)\} = 0$, hence by Lemma 3.13, there is $h_0 \in \mathcal{C}^\infty(K)$ such that $\mathcal{L}h_0 = 0$ and $h_0(0,y,t) = w(0,y,t)$, thus, (3.16) is verified when $k = 0$. Assume now that $h_1, \ldots, h_k$ have already been found in such a way that (3.16) is verified. Next we apply the operator ∂_x^{k+1} to $\mathcal{L}w_k$ and restrict to $x = 0$; we obtain, from (3.16), that $\mathcal{L}_0\{(\partial_x^{k+1}w_k)(0,y,t)\} = 0$. Lemma 3.13 again implies the existence of $h_{k+1} \in \mathcal{C}^\infty(K)$ such that $\mathcal{L}h_{k+1} = 0$ and $h_{k+1}(0,y,t) = (\partial_x^{k+1}w_k)(0,y,t)$. Hence

$$w_{k+1}(x,y,t) \doteq w_k(x,y,t) - h_{k+1}(x,y,t)\frac{x^{k+1}}{(k+1)!} = \mathrm{O}\big(|x|^{k+2}\big),$$

for $(\partial_x^j w_{k+1})(0,y,t) = 0$, if $j = 0, \ldots, k+1$. The construction of the sequence $\{h_j\}$ is complete. Finally, take a sequence $\sigma_j \in C_c^\infty$ in the variable x, with $\sigma_j \equiv 1$ in a neighborhood of the origin, in such a way that

$$R(x,y,t) = \sum_{j=0}^{\infty} \sigma_j(x)h_j(x,y,t)\frac{x^j}{j!}$$

defines a smooth function in a neighborhood of K. We have $\mathcal{L}R = 0$, hence the function $v \doteq w - R$ satisfies $\mathcal{L}v = f$; therefore, from (3.16), we obtain that v is flat at $x = 0$. □

Let f be as in system (3.15) and define $g \doteq \mathsf{X}v$, where v is given by Proposition 3.14. Since

$$\mathcal{L}g = \mathcal{L}\mathsf{X}v = \mathsf{X}\mathcal{L}v = \mathsf{X}f = 0,$$

our problem becomes the one of solving:

$$\begin{cases} \mathcal{L}w = 0, \\ \mathsf{X}w = g, \end{cases} \tag{3.17}$$

where g is flat at $x = 0$, with $\mathcal{L}g = 0$, in a neighborhood of K (in fact, the final solution to the system (3.15) will be $u = v - w$).

3.6. The semi-global first integral

Recall that we are studying the operator,

$$\mathcal{L} = \partial/\partial t + c(x, y, t)\partial/\partial y, \quad c = a + \mathrm{i}b, \tag{3.18}$$

in the open set $(-\delta, \delta) \times (-3M - 1, 3M + 1) \times \mathbb{T}^1$ under the assumption that $b \geqslant 0$ for all (x, y, t), and that there is $\epsilon > 0$ such that $b(x, y, t) > 0$ if $(x, y, t) \in (-\delta, \delta) \times (-3M - 1, 3M + 1) \times (-\epsilon, \epsilon)$.

Recall also that we have $\mathsf{X}c = 0$, where

$$\mathsf{X} = \partial/\partial y - x^m \partial/\partial x, \tag{3.19}$$

which implies that

$$c(0, y, t) = a(0, y, t) + \mathrm{i}b(0, y, t) \doteq a_\circ(t) + \mathrm{i}b_\circ(t) \doteq c_\circ(t). \tag{3.20}$$

Finally recall that

$$\int_0^{2\pi} b_\circ(t)\,\mathrm{d}t = 2\pi, \tag{3.21}$$

and set

$$\sup_{0\leqslant t\leqslant 2\pi} \left| \int_0^t a_\circ(t')\,\mathrm{d}t' \right| = M, \quad \int_0^{2\pi} a_\circ(t)\,\mathrm{d}t = \lambda. \tag{3.22}$$

Let $K = [-\frac{\delta}{2}, \frac{\delta}{2}] \times [-3M - \frac{\delta}{2}, 3M + \frac{\delta}{2}] \times \mathbb{T}^1$. Now Lemma 3.12 implies that there is $U \in C^\infty(K)$ such that

$$\mathcal{L}U = -c_y. \tag{3.23}$$

Since $c_y = x^m c_x$, Proposition 3.14 , yields a solution U satisfying

$$U(x, y, t) = \mathrm{O}\big(|x|^m\big). \tag{3.24}$$

If we define:

$$V(x, y, t) = \int_0^y \mathrm{e}^{U(x,y',t)}\,\mathrm{d}y' - \int_0^t c(x, 0, t')\mathrm{e}^{U(x,0,t')}\,\mathrm{d}t', \tag{3.25}$$

we obtain, after straightforward computations,

$$\mathcal{L}V = 0. \tag{3.26}$$

Now define:

$$\theta(x) = 2\pi \left(\int_0^{2\pi} -\mathrm{i}c(x, 0, t)\mathrm{e}^{U(x,0,t)}\,\mathrm{d}t \right)^{-1}. \tag{3.27}$$

After contracting δ, if necessary, we obtain $\theta \in C^\infty(-\delta, \delta)$; note that, in view of (3.20)–(3.22) and (3.24), we have $\theta(0) = \frac{2\pi}{2\pi - \mathrm{i}\lambda}$. Now the periodicity of U in the variable t implies that

$$\theta(x)V(x, y, 2\pi) - \theta(x)V(x, y, 0) = -2\pi\mathrm{i}.$$

Thus the function Z defined by,

$$Z(x,y,t)=\mathrm{e}^{\theta(x)V(x,y,t)}, \tag{3.28}$$

satisfies $Z\in C^\infty(K)$, and

$$\mathcal{L}Z=0, \qquad Z_y\neq 0. \tag{3.29}$$

We also have:

$$Z(0,y,t)=\mathrm{e}^{\theta(0)V(0,y,t)}, \tag{3.30}$$

with

$$\begin{aligned}\theta(0)V(0,y,t)&=\frac{2\pi}{2\pi-\mathrm{i}\lambda}\left(y-\int_0^t c_\circ(t')\,\mathrm{d}t'\right)\\&=\frac{2\pi}{4\pi^2+\lambda^2}(2\pi+\mathrm{i}\lambda)\left(y-\int_0^t c_\circ(t')\,\mathrm{d}t'\right)\\&=\frac{2\pi}{4\pi^2+\lambda^2}\left\{\left[2\pi\left(y-\int_0^t a_\circ(t')\,\mathrm{d}t'\right)+\lambda\int_0^t b_\circ(t')\,\mathrm{d}t'\right]\right.\\&\qquad\left.+\mathrm{i}\left[\lambda\left(y-\int_0^t a_\circ(t')\,\mathrm{d}t'\right)-2\pi\int_0^t b_\circ(t')\,\mathrm{d}t'\right]\right\}.\end{aligned}$$

Let $\gamma_{(x,y,t)}(s)=(x,\phi(s,y),s+t)$ be the integral curve of $\Re\mathcal{L}$ with initial point (x,y,t). Since $\gamma_{(0,y,0)}(s)=(0,y+\int_0^s a_\circ(t)\,\mathrm{d}t,s)$, we have:

$$Z\big(\gamma_{(0,y,0)}(s)\big)=\mathrm{e}^{\frac{2\pi}{2\pi-\mathrm{i}\lambda}(y-\mathrm{i}\int_0^s b_\circ(t)\,\mathrm{d}t)}. \tag{3.31}$$

From (3.30) we obtain $|Z|_y(0,y,t)>0$, and $(\arg Z)_y(0,y,t)=\frac{2\pi}{4\pi^2+\lambda^2}\cdot\lambda\neq 0$, if $\lambda\neq 0$. By contracting $\delta>0$, if necessary, we may assume that

$$|Z|_y>0 \quad\text{and}\quad (\arg Z)_y\neq 0, \quad \text{if } \lambda\neq 0, \tag{3.32}$$

(in particular, $\operatorname{sgn}[(\arg Z)_y]=\operatorname{sgn}(\lambda)$, if $\lambda\neq 0$) in $(-\delta,\delta)\times(-3M-1,3M+1)\times\mathbb{T}^1$.

In what follows, consider for each $\epsilon'\in(0,1)$,

$$\mathcal{A}(\epsilon')\doteq\left\{\zeta\colon \mathrm{e}^{-\frac{4\pi^2(2M-\lambda)}{4\pi^2+\lambda^2}}-\epsilon'<|\zeta|<\mathrm{e}^{\frac{8\pi^2M}{4\pi^2+\lambda^2}}+\epsilon'\right\}, \quad \text{if } \lambda\geqslant 0,$$

and

$$\mathcal{A}(\epsilon')\doteq\left\{\zeta\colon \mathrm{e}^{-\frac{8\pi^2M}{4\pi^2+\lambda^2}}-\epsilon'<|\zeta|<\mathrm{e}^{\frac{4\pi^2(2M+\lambda)}{4\pi^2+\lambda^2}}+\epsilon'\right\}, \quad \text{if } \lambda<0.$$

Also, for each $(x,y,t)\in(-\delta,\delta)\times(-2M-1,2M+1)\times(-\epsilon,\epsilon)$, let $\beta_{(x,\phi(2\pi,y),t)}(s)=(x,\psi(s,\phi(2\pi,y)),t)$ be the integral curve of $-\operatorname{sgn}(\lambda)\Im(\mathcal{L})$ with initial point $(x,\phi(2\pi,y),t)$. Furthermore, define the juxtaposition $\alpha_{(x,y,t)}(s)=\tilde\beta_{(x,\phi(2\pi,y),t)}*\tilde\gamma_{(x,y,t)}(s)$, where $\tilde\gamma_{(x,y,t)}(s)=\gamma_{(x,y,t)}{}_{|[0,2\pi]}(s)$, $\tilde\beta_{(x,\phi(2\pi,y),t)}=\beta_{(x,\phi(2\pi,y),t)}{}_{|[0,s_0]}$, and s_0 is such that $\beta_{(x,\phi(2\pi,y),t)}(s_0)=(x,y,t)$. Denotes by $D_{\alpha_{(x,y,t)}}$ the domain of the curve $\alpha_{(x,y,t)}$.

Lemma 3.15. *The equality $Z(\alpha_{(x,y,t)}(s_1))=Z(\alpha_{(x,y,t)}(s_2))$ holds if and only if one have $\alpha_{(x,y,t)}(s_j)=\gamma_{(x,y,t)}(s_j)$, $j=1,2$, and one of the following properties is verified*:

(i) $s_1=0$, $s_2=2\pi$, *and $\gamma_{(x,y,t)}$ is periodic* (*note that in this case $\tilde\beta_{(x,\phi(2\pi,y),t)}(s)\equiv(x,y,t)$*);
(ii) $0<s_1<s_2<2\pi$ *and $b\circ\gamma_{(x,y,t)}{}_{|[s_1,s_2]}(s)\equiv 0$.*

Proof. We write $Z(x,y,t)=|Z(x,y,t)|e^{i\arg Z(x,y,t)}$. Since $\mathcal{L}Z=0$ we have:

$$\begin{aligned}0&=|Z|_t e^{\mathrm{i}\arg Z}+|Z|\mathrm{i}(\arg Z)_t e^{\mathrm{i}\arg Z}+a|Z|_y e^{\mathrm{i}\arg Z}+a|Z|\mathrm{i}(\arg Z)_y e^{\mathrm{i}\arg Z}\\&\quad+\mathrm{i}b|Z|_y e^{\mathrm{i}\arg Z}+\mathrm{i}b|Z|\mathrm{i}(\arg Z)_y e^{\mathrm{i}\arg Z}\\&=e^{\mathrm{i}\arg Z}\big[\big(|Z|_t+a|Z|_y-b|Z|(\arg Z)_y\big)+\mathrm{i}\big(|Z|(\arg Z)_t+a|Z|(\arg Z)_y+b|Z|_y\big)\big].\end{aligned}$$

Thus

$$\begin{cases}|Z|_t+a|Z|_y-b|Z|(\arg Z)_y=0,\\|Z|(\arg Z)_t+a|Z|(\arg Z)_y+b|Z|_y=0,\end{cases}$$

hence $(\arg Z)_t+a(\arg Z)_y=-b|Z|_y/|Z|\leqslant 0$ and $|Z|_t+a|Z|_y=b|Z|(\arg Z)_y$. In particular,

$$\big(\arg Z\circ\gamma_{(x,y,t)}\big)'(s)=-b\big(\gamma_{(x,y,t)}(s)\big)\frac{|Z|_y(\gamma_{(x,y,t)}(s))}{|Z|(\gamma_{(x,y,t)}(s))}\leqslant 0\tag{3.33}$$

and

$$\big(|Z|\circ\gamma_{(x,y,t)}\big)'(s)=\big(b|Z|(\arg Z)_y\big)\circ\gamma_{(x,y,t)}.\tag{3.34}$$

On the other hand, (3.32) implies that, along $\beta_{(x,\phi(2\pi,y),t)}$, $\arg Z$ satisfies:

$$\big(\arg Z\circ\beta_{(x,\phi(2\pi,y)),t)}\big)'(s)=-\operatorname{sgn}(\lambda)\big[b(\arg Z)_y\big]\circ\beta_{(x,\phi(2\pi,y)),t)}<0.\tag{3.35}$$

Hence, from (3.33) and (3.35) we conclude that $Z\circ\alpha_{(x,y,t)}(D_{\alpha_{(x,y,t)}})$ is a closed curve which winds once around the origin in the clockwise sense; furthermore, we obtain that $Z\circ\tilde{\gamma}_{(x,y,t)}((0,2\pi))\cap Z\circ\tilde{\beta}_{(x,\phi(2\pi,y),t)}((0,s_0))=\emptyset$. By (3.35), $Z\circ\tilde{\beta}_{(x,\phi(2\pi,y),t)}$ is injective, and by (3.33) and (3.34), the equality $Z(\tilde{\gamma}_{(x,y,t)}(s_1))=Z(\tilde{\gamma}_{(x,y,t)}(s_2))$, with $s_1<s_2$, holds if and only if $b\circ\gamma_{(x,y,t)}{}_{|[s_1,s_2]}(s)\equiv 0$. $\square$

The continuous dependence of $\gamma_{(x,y,0)}$ with respect to the initial data implies that there exists $0<\sigma<\frac{\delta}{2}$ such that $K_\sigma=\bigcup_{\substack{x\in[-\sigma,\sigma]\\ y\in[-2M-\sigma,2M+\sigma]}}\alpha_{(x,y,0)}(D_{\alpha_{(x,y,0)}})$ is a compact subset of $(-\delta,\delta)\times(-3M-1,3M+1)\times\mathbb{T}^1$, and contains $\{0\}\times\{0\}\times\mathbb{T}^1$ (according to the discussion following (3.14)).

Lemma 3.16. *Let $u\in C^\infty(K_\sigma)$ with $\mathcal{L}u=0$. If $(x,y,0)\in K_\sigma$, then*

$$\partial_y\left\{\int_0^{2\pi}\big[u\cdot(Z_t+aZ_y)\big]\big(\gamma_{(x,y,0)}(s)\big)\,\mathrm{d}s-\int_0^{s_0}\big(u\cdot\operatorname{sgn}(\lambda)bZ_y\big)\big(\beta_{(x,\phi(2\pi,y)),0)}(s)\big)\,\mathrm{d}s\right\}=0.$$

Proof. In the proof of this lemma we will use the notation $\mathcal{L}=D_3+cD_2$. Since

$$\begin{aligned}\mathcal{L}(D_2Z)&=D_3D_2Z+cD_2D_2Z=D_2D_3Z+D_2(cD_2Z)-D_2cD_2Z\\&=D_2(\mathcal{L}Z)-D_2cD_2Z=-D_2cD_2Z,\end{aligned}$$

we have:

$$\begin{aligned}&\partial_y\big\{\big[u\cdot(D_3Z+aD_2Z)\big]\circ\gamma_{(x,y,0)}(s)\big\}\\&\quad=\partial_y\big\{\big[u\cdot(-\mathrm{i}bD_2Z)\big]\circ\gamma_{(x,y,0)}(s)\big\}\\&\quad=D_2(-\mathrm{i}b\cdot uD_2Z)\big(\gamma_{(x,y,0)}(s)\big)\cdot\partial_y\phi(s,y)\\&\quad=\big[-\mathrm{i}D_2b\cdot uD_2Z-\mathrm{i}b\cdot D_2(uD_2Z)\big]\big(\gamma_{(x,y,0)}(s)\big)\cdot\partial_y\phi(s,y)\\&\quad=\big[-\mathrm{i}D_2b\cdot uD_2Z+(D_3+aD_2)(uD_2Z)-\mathcal{L}(uD_2Z)\big]\big(\gamma_{(x,y,0)}(s)\big)\cdot\partial_y\phi(s,y)\\&\quad=\big[D_2a\cdot uD_2Z+(D_3+aD_2)(uD_2Z)\big]\big(\gamma_{(x,y,0)}(s)\big)\cdot\partial_y\phi(s,y).\end{aligned}\tag{3.36}$$

Recall that $\partial_s\phi(s,y) = a(x,\phi(s,y),s) = a(\gamma_{(x,y,0)}(s))$. Thus we have $\partial_s(\partial_y\phi(s,y)) = \partial_y(\partial_s\phi(s,y)) = D_2a(\gamma_{(x,y,0)}(s))\cdot\partial_y\phi(s,y)$, hence by (3.36), we obtain

$$\partial_y\big\{\big[u\cdot(D_3Z + aD_2Z)\big]\circ\gamma_{(x,y,0)}(s)\big\} = \partial_s\big[(u\cdot D_2Z)\big(\gamma_{(x,y,0)}(s)\big)\cdot\partial_y\phi(s,y)\big].$$

Therefore

$$\partial_y\left\{\int_0^{2\pi}\big[u\cdot(D_3Z + aD_2Z)\big]\big(\gamma_{(x,y,0)}(s)\big)\,\mathrm{d}s\right\} = u\cdot D_2Z\big(\gamma_{(x,y,0)}(s)\big)\cdot\partial_y\phi(s,y)\Big|_0^{2\pi}. \tag{3.37}$$

On the other hand, by a change $s\mapsto\psi(s,\phi(2\pi,y))$ we obtain:

$$-\int_0^{s_0}\big(u\cdot\mathrm{sgn}(\lambda)bD_2Z\big)\big(\beta_{(x,\phi(s,y),0)}(s)\big)\,\mathrm{d}s = \int_{\phi(2\pi,y)}^{y}(u\cdot D_2Z)(x,s,0)\,\mathrm{d}s.$$

Also,

$$\begin{aligned}&\partial_y\left\{\int_{\phi(2\pi,y)}^{y=\phi(0,y)}(u\cdot D_2Z)(x,s,0)\,\mathrm{d}s\right\}\\ &\quad = u\cdot D_2Z(x,y,0)\cdot\partial_y\phi(0,y) - u\cdot D_2Z\big(x,\phi(2\pi,y),0\big)\cdot\partial_y\phi(2\pi,y).\end{aligned} \tag{3.38}$$

Therefore, from (3.37) and (3.38) the result follows. $\square$

3.7. The representation of semi-global solutions

We will now proceed to a major step in the transformation of our problem.
Set $\tilde{K_\sigma} = \bigcup_{\substack{x\in[-\sigma,\sigma]\\ y\in[-M-\sigma,M+\sigma]}}\alpha_{(x,y,0)}(D_{\alpha_{(x,y,0)}})$.

Proposition 3.17. *Given $\sigma>0$, there exist $0<\sigma_0\leqslant\sigma$ and $0<\epsilon'<1$ such that*

$$(x,y,t)\in\tilde K_{\sigma_0} \implies Z(x,y,t)\in\mathcal{A}(\epsilon'),$$

and such that the following property is verified: if $u\in C^\infty(K_\sigma)$ satisfies $\mathcal{L}u=0$ then there exists a smooth function H defined in $[-\sigma_0,\sigma_0]$ and taking its values in the space of holomorphic functions on $\mathcal{A}(\epsilon')$ such that

$$u(x,y,t) = H\big(x,Z(x,y,t)\big),\quad \textit{if } (x,y,t)\in\tilde K_{\sigma_0}. \tag{3.39}$$

Proof. Let $u\in C^\infty(K_\sigma)$ such that $\mathcal{L}u=0$.

By using (3.30), (3.31), and uniform continuity, we reach the conclusion that there exist $\kappa,\rho_1,\rho_2,\rho_1',\rho_2'$, with $0<\kappa<\sigma$, $\rho_1>\mathrm{e}^{\frac{8\pi^2M}{4\pi^2+\lambda^2}}$ and $\rho_2<\mathrm{e}^{-\frac{4\pi^2(2M-\lambda)}{4\pi^2+\lambda^2}}$ if $\lambda\geqslant 0$, whereas $\rho_1'>\mathrm{e}^{\frac{4\pi^2(2M+\lambda)}{4\pi^2+\lambda^2}}$ and $\rho_2'<\mathrm{e}^{-\frac{8\pi^2M}{4\pi^2+\lambda^2}}$ if $\lambda<0$, such that for $|x|\leqslant\kappa$ and $s\in D_\alpha$,

$$\big|Z\big(\alpha_{(x,2M+\sigma,0)}(s)\big)\big|\geqslant\rho_1,\qquad \big|Z\big(\alpha_{(x,-2M-\sigma,0)}(s)\big)\big|\leqslant\rho_2,\quad \text{if } \lambda\geqslant 0, \tag{3.40}$$

and

$$\big|Z\big(\alpha_{(x,2M+\sigma,0)}(s)\big)\big|\geqslant\rho_1',\qquad \big|Z\big(\alpha_{(x,-2M-\sigma,0)}(s)\big)\big|\leqslant\rho_2',\quad \text{if } \lambda<0. \tag{3.41}$$

Indeed, from (3.30) and (3.31), we obtain:

$$\big|Z\circ\beta_{(0,\phi(2\pi,y),0)}(s)\big| = \mathrm{e}^{\frac{4\pi^2\psi(s,\phi(2\pi,y))}{4\pi^2+\lambda^2}}$$

and

$$\big|Z\big(\gamma_{(0,y,0)}(s)\big)\big| = \mathrm{e}^{\frac{2\pi}{4\pi^2+\lambda^2}(2\pi y+\lambda\int_0^s b_\circ(t)\,\mathrm{d}t)},$$

hence, if $\lambda \geqslant 0$, then

$$\left|Z\left(\alpha_{(0,2M+\sigma,0)}(s)\right)\right| \geqslant \mathrm{e}^{\frac{4\pi^2(2M+\sigma)}{4\pi^2+\lambda^2}},$$

and

$$\left|Z\left(\alpha_{(0,-2M-\sigma,0)}(s)\right)\right| \leqslant \mathrm{e}^{-\frac{4\pi^2(2M+\sigma-\lambda)}{4\pi^2+\lambda^2}};$$

whereas if $\lambda < 0$, then

$$\left|Z\left(\alpha_{(0,2M+\sigma,0)}(s)\right)\right| \geqslant \mathrm{e}^{\frac{4\pi^2(2M+\sigma+\lambda)}{4\pi^2+\lambda^2}},$$

and

$$\left|Z\left(\alpha_{(0,-2M-\sigma,0)}(s)\right)\right| \leqslant \mathrm{e}^{-\frac{4\pi^2(2M+\sigma)}{4\pi^2+\lambda^2}}.$$

By Lemma 3.16 the following functions of $x \in [-\kappa, \kappa]$ are well-defined and smooth, for $j \in \mathbb{Z}$:

$$\begin{aligned} A_j(x) = &-\frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{u(\gamma_{(x,y,0)}(s))}{Z(\gamma_{(x,y,0)}(s))^{j+1}} (Z_t + aZ_y)\left(\gamma_{(x,y,0)}(s)\right) \mathrm{d}s \\ &- \frac{1}{2\pi \mathrm{i}} \int_0^{s_0} \frac{u(\beta_{(x,\phi(2\pi,y),0)}(s))}{Z(\beta_{(x,\phi(2\pi,y),0)}(s))^{j+1}} \left(-\operatorname{sgn}(\lambda) bZ_y\right)\left(\beta_{(x,\phi(2\pi,y),0)}(s)\right) \mathrm{d}s. \end{aligned}$$

From (3.40) and (3.41) we obtain, for $x \in [-\kappa, \kappa]$, the following estimates, for $j \neq 0$:

$$\left|A_j^{(\ell)}(x)\right| \leqslant C_\ell |j|^\ell \rho_2^{|j|}, \quad j \leqslant -1 \qquad \text{and} \qquad \left|A_j^{(\ell)}(x)\right| \leqslant C_\ell |j|^\ell \rho_1^{-|j|}, \quad j > -1, \quad \text{if } \lambda \geqslant 0;$$

and

$$\left|A_j^{(\ell)}(x)\right| \leqslant C_\ell |j|^\ell {\rho_2'}^{|j|}, \quad j \leqslant -1 \qquad \text{and} \qquad \left|A_j^{(\ell)}(x)\right| \leqslant C_\ell |j|^\ell {\rho_1'}^{-|j|}, \quad j > -1, \quad \text{if } \lambda < 0.$$

Therefore, for some $\epsilon' > 0$, the Laurent series

$$H(x, \zeta) = \sum_{j \in \mathbb{Z}} A_j(x) \zeta^j \tag{3.42}$$

defines a smooth function of $x \in [-\kappa, \kappa]$ valued in $\mathcal{O}(\mathcal{A}(\epsilon'))$. Note that ϵ' depends only on ρ_1, ρ_2 (respectively ρ_1', ρ_2').

Let us now fix $0 < \sigma_0 < \kappa$ such that

$$(x, y) \in [-\sigma_0, \sigma_0] \times [-M - \sigma_0, M + \sigma_0] \Rightarrow Z\left(\alpha_{(x,y,0)}(D_{\alpha_{(x,y,0)}})\right) \subset \mathcal{A}(\epsilon').$$

We must prove (3.39). Fix $(x, y, 0) \in \tilde{K}_{\sigma_0}$. Note that

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{H(x, Z(\gamma_{(x,y,0)}(s)))}{Z(\gamma_{(x,y,0)}(s))^{j+1}} (Z_t + aZ_y)\left(\gamma_{(x,y,0)}(s)\right) \mathrm{d}s + \frac{1}{2\pi \mathrm{i}} \int_{\phi(2\pi,y)}^{y} \frac{H(x, Z(x,s,0))}{Z(x,s,0)^{j+1}} Z_y(x,s,0)\, \mathrm{d}s \\ &\quad = \frac{1}{2\pi \mathrm{i}} \int_{Z \circ \alpha_{(x,y,0)}} \frac{H(x,\zeta)}{\zeta^{j+1}} \mathrm{d}\zeta = -A_j(x). \end{aligned}$$

Thus, for each $j \in \mathbb{Z}$, we have:

$$\begin{aligned} &\int_0^{2\pi} \left[u\left(\gamma_{(x,y,0)}(s)\right) - H\left(x, Z\left(\gamma_{(x,y,0)}(s)\right)\right)\right] \frac{(Z_t + aZ_y)(\gamma_{(x,y,0)}(s))}{Z(\gamma_{(x,y,0)}(s))^{j+1}} \mathrm{d}s \\ &\quad + \int_{\phi(2\pi,y)}^{y} \left[u(x,s,0) - H\left(x, Z(x,s,0)\right)\right] \frac{Z_y(x,s,0)}{Z(x,s,0)^{j+1}} \mathrm{d}s = 0. \end{aligned} \tag{3.43}$$

There is a continuous function $u^\sharp : Z(\alpha_{(x,y,0)}(D_{\alpha_{(x,y,0)}})) \to \mathbb{C}$ defined by $u(\alpha_{(x,y,0)}(s)) = u^\sharp(Z(\alpha_{(x,y,0)}(s)))$. Indeed, the function $u^\sharp$ is well-defined in view of Lemma 3.15, because $\mathcal{L}u = 0$; the continuity is trivial.

Set $v^\sharp = u^\sharp - H(x,\cdot)$; then (3.43) implies that

$$\int\limits_{Z\circ\alpha_{(x,y,0)}} \frac{v^\sharp(\zeta)}{\zeta^{j+1}}\,\mathrm{d}\zeta = 0, \quad \forall j \in \mathbb{Z}, \tag{3.44}$$

hence

$$\int\limits_{Z\circ\alpha_{(x,y,0)}} \frac{v^\sharp(\zeta)}{z-\zeta}\,\mathrm{d}\zeta = 0, \quad z \notin Z\circ\alpha_{(x,y,0)}. \tag{3.45}$$

It follows from [24] (Lemma 2.7) that $v^\sharp = 0$. The proof of (3.39) is complete. □

3.8. End of the proof of Theorem 3.4

We are now ready to complete the resolution of the system in (3.17), that is, $\mathcal{L}w = 0$, $\mathsf{X}w = g$; we will use $K = K_\sigma$. Recall that $\mathcal{L}g = 0$; thus Proposition 3.17 yields:

$$g(x,y,t) = G\big(x, Z(x,y,t)\big),$$

where G is a smooth function on $[-\eta,\eta]$, valued in $\mathcal{O}(\mathcal{A}(\epsilon'))$, for some $\eta, \epsilon' > 0$; furthermore, G is flat at $x = 0$.

We will look for a solution, w, to (3.17), with w of the form,

$$w(x,y,t) = W\big(x, Z(x,y,t)\big),$$

where $W \in C^\infty((-\eta',\eta'), \mathcal{O}(\mathcal{A}(\epsilon)))$, for some η'. We automatically have $\mathcal{L}w = 0$ and all we need now is to guarantee that $\mathsf{X}w = g$.

If w is of the above form, then we have:

$$\begin{aligned}
\mathsf{X}w(x,y,t) &= \big(\partial_y - x^m\partial_x\big)\big[W\big(x, Z(x,y,t)\big)\big] \\
&= W_\zeta\big(x, Z(x,y,t)\big)Z_y(x,y,t) - x^m\big[W_x\big(x, Z(x,y,t)\big) + W_\zeta\big(x, Z(x,y,t)\big)Z_x(x,y,t)\big] \\
&= (\mathsf{X}Z)(x,y,t)W_\zeta\big(x, Z(x,y,t)\big) - x^m W_x\big(x, Z(x,y,t)\big).
\end{aligned}$$

At this point it is important to remark that we have $\mathcal{L}(\mathsf{X}Z) = \mathsf{X}(\mathcal{L}Z) = 0$, hence we may apply Proposition 3.17 to write (by contracting η and ϵ' if necessary),

$$(\mathsf{X}Z)(x,y,t) = A\big(x, Z(x,y,t)\big),$$

where A is a smooth function on $[-\eta,\eta]$, valued in $\mathcal{O}(\mathcal{A}(\epsilon'))$, for some $\eta, \epsilon' > 0$.

Our computation above yields:

$$\mathsf{X}w(x,y,t) = A\big(x, Z(x,y,t)\big)W_\zeta\big(x, Z(x,y,t)\big) - x^m W_x\big(x, Z(x,y,t)\big).$$

Hence we have $\mathsf{X}w = g$ if and only if we have:

$$A\big(x, Z(x,y,t)\big)W_\zeta\big(x, Z(x,y,t)\big) - x^m W_x\big(x, Z(x,y,t)\big) = G\big(x, Z(x,y,t)\big),$$

or, with $\zeta = Z(x,y,t)$,

$$A(x,\zeta)W_\zeta(x,\zeta) - x^m W_x(x,\zeta) = G(x,\zeta).$$

In other words, we are led to finding a solution, W, to the equation,

$$\mathcal{P}W = G, \tag{3.46}$$

where

$$\mathcal{P} = A(x,\zeta)\partial/\partial\zeta - x^m\partial/\partial x. \tag{3.47}$$

An analysis of the above computations shows that the following proposition is true (see also [1], Proposition 6.2).

Proposition 3.18. *Suppose that for some $0 < \eta' \leqslant \eta$ there exists a solution, $W \in C^\infty((-\eta', \eta'), \mathcal{O}(\mathcal{A}(\epsilon')))$, to* (3.46)*, where $\mathcal{P}$ is given by* (3.47)*. Then we have that the function $w(x,y,t) = W(x, Z(x,y,t))$ is a solution to* (3.17) *in a neighborhood of* $\{0\} \times \{0\} \times \mathbb{T}^1$.

Next, as in [1], we study the function $A(x,\zeta)$. We have:

$$A\big(x, Z(x,y,t)\big) = (\mathsf{X}Z)(x,y,t) = \mathsf{X}\big\{\theta(x)V(x,y,t)\big\}Z(x,y,t),$$

and

$$\mathsf{X}\big\{\theta(x)V(x,y,t)\big\} = \theta(x)\partial_y V - x^m V\partial_x\theta - x^m\theta\partial_x V = \theta(x)\mathrm{e}^{U(x,y,t)} + \mathrm{O}\big(|x|^m\big),$$

therefore

$$\mathsf{X}\big\{\theta(x)V(x,y,t)\big\} - \theta(x) = \theta(x)\big\{\mathrm{e}^{U(x,y,t)} - 1\big\} + \mathrm{O}\big(|x|^m\big);$$

hence (3.24) implies that

$$\mathsf{X}\big\{\theta(x)V(x,y,t)\big\} - \theta(x) = \mathrm{O}\big(|x|^m\big).$$

We obtain:

$$A\big(x, Z(x,y,t)\big) = \mathsf{X}\big\{\theta(x)V(x,y,t)\big\}Z(x,y,t) = \theta(x)Z(x,y,t) + \mathrm{O}\big(|x|^m\big). \tag{3.48}$$

Define:

$$A^\sharp(x,\zeta) = A(x,\zeta) - \theta(x)\zeta. \tag{3.49}$$

Now (3.48) implies

$$\big(\partial_x^j A^\sharp\big)\big(0, Z(0,y,t)\big) = 0, \quad j = 0,\dots,m-1,$$

hence

$$A^\sharp(x,\zeta) = \mathrm{O}\big(|x|^m\big). \tag{3.50}$$

(here we use the fact that $A^\sharp(x,\zeta)$ is holomorphic in ζ and $\partial_x^j A^\sharp(0,\zeta) = 0$, if $\zeta = Z(0,y,t)$, for $j = 0,\dots,m-1$).

As in [1] (pp. 22–23), one sees that, in order to solve (3.46) it suffices to solve the equation:

$$x^m w'(x) = \theta(x)Tw(x) + x^m\big(B(x)Tw(x) + g(x)\big), \tag{3.51}$$

where $m \geqslant 2$, $T \doteq \zeta\partial_\zeta$, $B(x) = B(x,\zeta) = \frac{A^\sharp(x,\zeta)}{x^m\zeta}$ and $g : [-\eta,\eta] \to \mathrm{H}$, where $\mathrm{H} = \mathcal{O}(\mathcal{A}(\epsilon')) \cap L^2(\mathcal{A}(\epsilon'))$ (g is a given smooth function which is flat at $x = 0$); also, (3.51) satisfies the hypotheses of Theorem A.1.1 of [1]. Since Theorem A.1.1 of [1] solves (3.51), the proof is complete. $\square$

References

[1] A.P. Bergamasco, P.D. Cordaro, G. Petronilho, Global solvability for a class of complex vector fields on the two-torus, Comm. Partial Differential Equations 29 (2004) 785–819.

[2] A.P. Bergamasco, A. Meziani, Semiglobal solvability of a class of planar vector fields of infinite type, Mat. Contemporânea 18 (2000) 31–42.

[3] A.P. Bergamasco, A. Meziani, Solvability near the characteristic set for a class of planar vector fields of infinite type, Ann. Inst. Fourier (Grenoble) 55 (2005) 77–112.

[4] A.P. Bergamasco, W.V.L. Nunes, S.L. Zani, Global properties of a class of overdetermined systems, J. Funct. Anal. 200 (2003) 31–64.

[5] A.P. Bergamasco, P.L. Dattori da Silva, Global solvability for a special class of vector fields on the torus, Contemp. Math. 400 (2006) 11–20.

[6] S. Berhanu, G.A. Mendoza, Orbits and global unique continuation for systems of vector fields, J. Geom. Anal. 7 (1997) 173–194.

[7] P.D. Cordaro, X. Gong, Normalization of complex-valued planar vector fields which degenerate along a real curve, Adv. Math. 184 (2004) 89–118.

[8] D.B.A. Epstein, Curves on 2-manifolds and isotopies, Acta Math. 115 (1966) 83–107.

[9] P. Godin, Propagation des singularités pour les opérateurs différentiels de type principal, localement résolubles, à coefficients analytiques, en dimension 2, Ann. Inst. Fourier (Grenoble) 29 (2) (1979) 223–245.

[10] B. Helffer, Addition de variables et applications à la régularité, Ann. Inst. Fourier (Grenoble) 28 (2) (1978) 221–231.

[11] M.W. Hirsch, Differential Topology, Graduate Texts in Mathematics, vol. 33, Springer, New York, 1976.

[12] L. Hörmander, Propagation of singularities and semi-global existence theorems for (pseudo-) differential operators of principal type, Ann. Math. 108 (1978) 569–609.

[13] L. Hörmander, Pseudo-differential operators of principal type, in: NATO Advanced Study Inst. on Singularities in Boundary Value Problems, Reidel, Dordrecht, 1981, pp. 69–96.

[14] L. Hörmander, The Analysis of Linear Partial Differential Operators IV, Springer-Verlag, New York, 1984.

[15] J. Hounie, Globally hypoelliptic and globally solvable first order evolutions equations, Trans. Amer. Math. Soc. 252 (1979) 233–248.

[16] J. Hounie, Minimal sets of families of vector fields on compact surfaces, J. Diferential Geom. 16 (1981) 739–744.

[17] J. Hounie, Globally Hypoelliptic Vector Fields on Compact Surfaces, Comm. Partial Differential Equations 7 (1982) 343–370.

[18] J. Hounie, A note on global solvability of vector fields, Proc. Amer. Math. Soc. 94 (1985) 61–64.

[19] G. Köthe, Topological Vector Spaces II, Springer-Verlag, New York, 1979.

[20] A. Meziani, Elliptic planar vector fields with degeneracies, Trans. Amer. Math. Soc. 357 (2005) 4225–4248.

[21] H.J. Sussmann, Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc. 180 (1973) 171–188.

[22] A. Schwartz, A generalization of the Poincaré–Bendixson theorem to closed two-dimensional manifolds, Amer. J. Math. 85 (1963) 453–458.

[23] F. Treves, Approximation and Representation of Functions and Distributions Annihilated by a System of Complex Vector Fields, École Polytechnique, Centre de Mathématiques, Palaiseau, 1981.

[24] J. Wermer, Banach Algebras and Several Complex Variables, second ed., Springer-Verlag, Berlin, 1976.